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AN

ELEMENTARY TREATISE

ON

ALGEBRA,

THEORETICAL AND PRACTICAL.

BY

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PREFACE.

In the following Treatise, it has been the aim of the Author to render the various investigations and processes as simple, natural, and easy as possible, and to establish and illustrate the principles in a plain and familiar manner. He trusts, in particular, that the modes in which he has established the rule for subtraction (not new, he admits, in principle), and the rule for the signs in multiplication and division, and also the methods which he has employed in explaining the theory of fractions and radicals, will divest those subjects of much of the mystery and difficulty which have unnecessarily been thrown around them. In a similar manner, using freely, but not servilely, what has been done by previous writers, he has endeavoured to simplify the elementary principles and processes in the resolution of equations; and he hopes that the views and explanations which he has given regarding infinite quantities and the sums of infinite series, will remove much of the difficulty which is very generally and very naturally felt in reference to those subjects. Of the binomial theorem, he has given a proof which he considers simple and easy; and the experienced algebraist will recognise various other improvements in the course of the work, which it is unnecessary here to particularise.

Much absolute novelty, unless in the mode of exposition, cannot now be expected in a work on algebra. Some things, however, the Author believes are new altogether, while others are now for the first time fully developed, and extensively and advantageously applied. Of this kind is the method of *detached coefficients*, which is explained and exemplified in pages 26. 35. 84. &c.; a method which is at once greatly shorter and easier than the common one, and which, throughout the operation, relieves the mind from the trouble and fatigue of considering what powers of the quantities are to be written in the several terms. It prepares the student also for many of the investigations connected with the theory and resolution of equations, and it is employed with advantage on many other occasions. He believes also, that the methods of resolving quadratic equations by means of the rules in pages 135. and 139. will be found to be much superior, in point of facility and simplicity, to any other method that has yet appeared; as those rules preclude the necessity of "completing the square," extracting the square root, and transposing; and they prevent all trouble in the management of fractions; while, in the common method, the fractional operations often constitute the principal part of the labour.* In page 63. there is given a new method of finding the greatest common divisor, by means of which the required result is often obtained much more easily than by the common method, according to the rule in page 59. In pages 49. and 73., the advantage of employing negative indices in certain cases, so as to prevent the necessity of performing fractional operations, is pointed out and exemplified; and a new and easy method of "proving" operations in multiplication and division is given in the notes in pages 24. and 33.

In Chapter X. a brief outline of some of the principal subjects comprehended in the theory of equations is given; and, along with it, the beautiful method of resolving numerical equations, discovered by the late Mr. Horner, is exhibited at considerable length, and in a mode which it is hoped will be found to be very simple and easy, both in its theory, and in practice. That the brief introduction to the theory of equations here given must be very imperfect will be readily anticipated, if it be considered that Professor Young of Belfast has published a work in five hundred octavo pages on the *Theory and Solution of Algebraical Equations of the higher Orders.* What is here given on this curious and interesting branch of analysis will perhaps be sufficient for the greater number of mathematical students; but those who may wish to obtain a more extensive knowledge of the subject, and to devote the time and labour necessary for accomplishing that object, may have recourse to Mr. Young's work above referred to.

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* The first of these rules was published by the Author in a Belfast periodical in 1825, and the second occurred to him at a later period. A particular case of the second rule is given by Bonnycastle in his larger *Algebra.* The *Bija Ganita,* a Hindoo work on algebra of the twelfth or thirteenth century, contains a method of solution which is much superior, in many instances, to the one which is commonly employed in Europe, as it prevents fractions from arising in the course of the operation. It requires, however, the formal completing of the square the extraction of the square root, &c. The rules here given combine the advantages of both these methods, and are free from their disadvantages, giving in every case the values of the unknown quantity without any intermediate work, and always in the simplest and best form. It is strange, that, while the modern analysts have so zealously and successfully exerted themselves in deriving easy rules for practical purposes in numberless cases, they should not have thought of establishing similar rules for the solution of quadratic equations, but should have gone on in the old way, always dividing by the coefficient of the highest power, completing the square, extracting the square root, and transposing.

The examples and exercises are, in the great majority of instances, new. Several, however, have been taken from foreign, and particularly German, works; and some are from English writers of past times, but none from living English authors, except in a few instances in which improved solutions are given or indicated. Several instances of this kind will be found in the Chapter on the Diophantine Analysis.

As to the mode of employing the work as a text-book, the teacher must be guided mainly by his own judgment. It may be remarked, however, that in a first course, unless the pupil possess more than ordinary ability, the more difficult parts may be omitted, and may be taken up afterwards, when the learner has acquired more power in the management of algebraic quantities, and in the performance of operations. Thus, there are portions of Chapter III., which, however important and valuable, will at first be found by most learners to be somewhat abstract, and not so easily followed out as many subsequent parts of the work, and which may therefore be postponed. For the same reason, several things regarding fractions, radicals, involution, evolution, and some other subjects, may at first be omitted with advantage; and the same may be done, with great propriety, regarding several of the more difficult exercises in the various Chapters. What has now been said will be applicable, in a still greater degree, in reference to those who may study the work without the aid of a teacher.

The learner, whether he has a teacher or not, should by no means be dispirited by feeling at first some difficulty in the study of algebra. Few studies are free at their commencement from something that is unattractive, or even forbidding: and though in algebra, from its peculiar symbolical language, and its abstract and general character, some difficulty will often be felt for a short time, this will soon be removed by steady application, and by continued practice; and the student, on returning to investigations or exercises which, when first attempted, had surpassed his powers, will often be at a loss to discover how they could formerly have presented any difficulty. He ought to be encouraged, also, by reflecting on the extreme value of the science. In itself, indeed, and in its application and extension in the differential and integral calculus, and in other branches of pure mathematics, it is a most powerful, an indispensable, instrument for prosecuting investigations in mechanics, astronomy, and other subjects in physical science; and, without its aid, it is impossible to understand, or duly to appreciate, the discoveries of Newton, Laplace, and the other great men who have done such wonders in extending the boundaries of modern science.

From the compressed manner in which the work is printed, it contains a very large amount of matter in proportion to its size. At the same time, the Author has been obliged to omit a few subjects (not, indeed, of primary importance), which he would have liked to insert, had it not been that both he and the Publishers wished to supply a useful and comprehensive elementary work at as moderate a price as possible. The student, however, who shall make himself acquainted with what is here given, will feel no difficulty in reading any thing additional that will be found in the more extended treatises on the subject.

Throughout the work, the Author has carefully kept clear of every thing of a metaphysical or disputed character. With regard to all the practically useful applications and interpretations of algebra, there is no difference of opinion among men of sound science and judgment; and it is only with such matters that the mere learner should have any concern. Those who may wish to know something of the other subjects referred to, such as what has been called "symbolical algebra," and the interpretation of imaginary quantities, may have recourse to Peacock's *Algebra*, or to his Paper in the *Report of the British Association for* 1833; and to a very acute and sensible article, controverting Dr. Peacock's views, which will be found in the form of a note at the end of Bryce's *Algebra*.

Glasgow College, May 18. 1844.

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AN

ELEMENTARY TREATISE

ON

ALGEBRA.

CHAPTER I.

DEFINITIONS AND ELEMENTARY PRINCIPLES.*

1. THE term *quantity* is used in mathematics to denote whatever is, or may be, expressed by means of a number. Thus, 5 days, 8 gallons, 20 acres, and a line of any length are all quantities.

2. *Algebra* is the science in which investigations regarding mathematical quantities are conducted by means of general symbols or characters. †

3. The symbols employed in algebra are used to denote (1.) the quantities under consideration; (2.) the comparative magnitudes of quantities; and (3.) the operations which are indicated as being performed upon them.

4. The letters of the alphabet have been adopted, by common consent, as the first of those classes of characters: and, in the so-

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* See Note A. at the end of the volume.

† In common arithmetic, characters are also employed as well as in algebra: in it, however, each character has merely one signification. Thus, 5 always represents the number five, such as five hundred, five thousand (in the expressions 500 and 5000), five men, five days, &c. In algebra, on the contrary, the symbols representing quantities may have an infinite variety of significations. Thus, in an investigation regarding a triangle, the letter a may be employed to denote a side, whether that side be 6 inches, 10 yards, or 100 miles in length. In an historical point of view, algebra is an extension and generalisation of common arithmetic; and hence it has been called *universal arithmetic*. This definition has been regarded by many as too limited for the science in its present state. It is impossible, indeed, to give any definition that will convey an adequate idea of the nature of the science to those who are unacquainted with it; and it is, perhaps, equally impossible to give one that would be generally satisfactory to algebraists themselves, nor is it necessary.

lution of problems, it has been the practice of the later mathematicians to express the given or known quantities by the first letters, a, b, c, &c.; and the unknown by the last, such as x, y, z. When only the solution of a particular question is required, the known numbers are often expressed by means of the common notation, 1, 2, 3, 100, &c. When letters are so used, each of them may be regarded as expressing a certain number of the units concerned in the subject of inquiry. Thus, suppose an investigation regarded distance in miles, then a would represent a certain number, such as 10, or 20, of those miles; a mile being the unit.*

5. The comparative magnitudes of quantities are denoted by placing between them one of the characters or *signs*, $=$, $>$, $<$; the first of which is read *equal to*, or simply *equal;* the second, *greater than;* and the third, *less than:* the opening in each of the two latter being turned towards the greater quantity. Thus, $a=12$, $a>10$, and $a<20$ express respectively, that a is equal to 12, that a is greater than 10, and that a is less than 20. The marks of proportion, : :: :, serve likewise to express the relative magnitudes of quantities.

6. The operations of addition and subtraction are expressed by the characters or signs, $+$, called *plus*, and $-$, called *minus;* the former denoting, that the quantity after it is to be *added* to the quantity or quantities with which it is, or may afterwards be, connected; while the latter signifies, that the quantity which follows it, is to be *subtracted* from the quantity or quantities with which it is now, or with which it may afterwards be, connected. Thus, $5+3=8$ is read, *5 plus 3 equal to* 8; that is, *5 more by 3* equal to 8: while $9-3=6$ is read, *9 minus 3 equal to 6;* that is, *9 less by 3 equal to* 6. A quantity which is preceded by $-$, is said to be *negative;* while one which has $+$, or neither $+$ nor $-$ before it, is called a *positive* quantity, or, by the older writers, an *affirma-*

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* In investigations, when there are two or more quantities which have any peculiar relation to each other, such as when each is constantly derivable from the one preceding it by the same process, it is often of much advantage to represent them by the same letter, marked, for the sake of distinction, with accents, or with numbers or letters placed, commonly, to the right of them, and lower than the general line; thus, x', x'', &c., or x_1, x_2, x_3, x_n, &c. This notation will be frequently employed hereafter, particularly in the sections on Series and Equations. For brevity, x', x'', &c., are generally read, *x dash*, *x double dash*, &c.; and x_1, x_2, &c., may be read, *x one*, *x two*, &c., or, *first x*, *second x*, &c. We shall see in § 11. that the notation for one most important class of quantities differs in some degree from what is here pointed out. (The mark § refers to the numbering of the paragraphs.)

tive one. When two or more quantities are added together, the result is termed their *sum;* but if one quantity be taken from another, the remainder is called their *difference.*

7. The multiplication of quantities expressed by letters, is generally denoted by writing the letters in succession, as in a word. Thus, ab, which for brevity is read a *into* b, or simply ab, denotes the result obtained by multiplying the number expressed by a by the one expressed by b; so that, if 10 were denoted by a, and 5 by b, ab would be equal to 50. The result ab is called the *product* of a and b; and a and b the *factors* of that product. In like manner, abc denotes the result obtained by multiplying the product of a and b by c; so that if a were equal to 2, b to 3, and c to 4, abc would be equal to 24. In this and similar cases, in which there are more than two factors, the result is called the *continual* or *continued product* of those factors. On the same principle of notation, the multiplication of a quantity by a number, is expressed by prefixing the number to the quantity. Thus, $4a$ and $5bc$ denote respectively four times a and five times bc.

In some cases, multiplication is expressed by interposing a point, or the sign ×, between the factors. Thus, the product of 2 and 3 is denoted by 2×3; and the continual product of 2, 3, and 4, by $2.3.4$, or by $2 \times 3 \times 4$. These modes of denoting multiplication are seldom used, except when the factors are numbers expressed in the common notation; in which case the simple writing of the factors in succession is obviously inadmissible; the pre-established meaning, for instance, of 234 (two hundred and thirty-four) being very different from the continual product of 2, 3, and 4.

8. Division is generally expressed by writing the dividend over the divisor with a line between them, after the manner of a fraction. The same operation is sometimes denoted by writing the dividend before the sign ÷ or :, and the divisor after it. Thus, $\frac{a}{b}$, $a \div b$, or $a : b$, each of which is read briefly a *by* b, signifies the quotient obtained by dividing a by b. We shall hereafter see a mode of expressing division different from both of these.

9. The number, whether whole or fractional, which is placed before a quantity, as a multiplier, is often called its *coefficient.* Thus, in the expressions, $3a$, na, the multipliers 3 and n are coefficients of a: the first a *numeral* or *numerical* one; and the second, being expressed by a letter, a *literal* one. In like manner, in the expression $\frac{3}{4}x$, the coefficient of x is $\frac{3}{4}$. When no coefficient is written, 1 is to be understood. Thus, a is the same as $1a$.

10. When a quantity is employed any number of times as factor, the product is called a *power* of that quantity; the *second power*, if the quantity be used twice as factor; the *third*, if three times; the *fourth*, if four times; and in general, the *nth* power, if n times, n being any whole positive number: and, in reference to any power of a quantity, the quantity itself is called the *root* of that power. Thus, aa, aaa, $aaaa$, $aaaaa$, are respectively the second, third, fourth, and fifth powers of a; and a is called the second root of aa, the third root of aaa, the fourth root of $aaaa$, &c. So also, the second, third, fourth, and fifth powers of 3 are 9, 27, 81, and 243; while 3 is the second root of 9, the third root of 27, the fourth root of 81, and the fifth root of 243. The second power of a quantity is generally called its *square*, and the third power its *cube* *: and in like manner, its second and third roots are commonly called its *square* and *cube* roots. The fourth power of a quantity is sometimes called its *biquadrate*, and the fourth root its *biquadratic root*. These latter terms, however, are properly falling into disuse.

Hence it appears, that the second or square root of a quantity is that quantity which, used twice as factor, will produce the original quantity; the third or cube root, that which, employed three times as factor, will produce the same quantity; and so on. Thus, the square root of 9 is 3, because $3 \times 3 = 9$; and the cube root of 64 is 4, because $4 \times 4 \times 4 = 64$.

The square root of a quantity is commonly denoted by prefixing to it the sign $\sqrt{}$. Thus, $\sqrt{100}$ and $\sqrt{a}$ indicate respectively the square root of 100 and the square root of a. In like manner, $\sqrt[3]{100}$ denotes the third or cube root of 100; $\sqrt[4]{100}$ its fourth root, &c. Another, and, in general, a preferable, notation for roots will be explained in a subsequent section.

We shall see hereafter, that there are important extensions of the simple and primitive meaning of powers here given.

11. The notation for powers employed in the last § would

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* These very improper names for the second and third powers, had their origin in the circumstance, that the area of a square is computed by multiplying the number expressing the length of one of its sides by itself, while the content of a cube is found by multiplying the number expressing the length of a side of its base by itself, and the product by the same number. The terms, *square* root and *cube* root, applied to abstract numbers, are equally improper. All these terms, however, as well as several other inappropriate ones, have got to be so universally employed, that their use is not likely to be discontinued. The term *biquadrate* signifies that the operation of squaring is *twice* performed.

evidently be very inconvenient, especially for the higher ones. To obviate this, the root is written but once, and, to the right of it, and a little higher, a number is placed, showing how often the root is employed as factor, and thus pointing out the order of the power. Thus, aa, aaa, $aaaa$, and $aaaaa$, are written a^2, a^3, a^4, a^5; and, in general, if n be put to denote the number of times a is employed as factor, we shall have the nth power of a, that is, $aaaa\ldots\ldots$, a being repeated n times, expressed by a^n. These expressions, a^2, a^3, a^4, a^5, a^n, are therefore *powers* of a; and the numbers, 2, 3, 4, 5, and n, are called the *indices* or *exponents* of those powers. *

By assigning different positive whole values to n, it will be seen that a^n is a general expression, comprehending all such powers as we have been considering. Thus, by taking it successively equal to 2, 3, 4, and 5, we get a^2, a^3, a^4, and a^5, the particular powers above considered: and if we take n equal to 1, we have a^1, which, as we shall see hereafter, is the same as a itself.

12. A quantity which consists of two or more quantities combined by addition or subtraction is called a *compound quantity*. A quantity which is not so composed is called a *simple quantity*. Thus 5 and a are simple quantities; as are also $3bx$ and $\frac{1}{2}y^2z$: while $5+a-3bx+\frac{1}{2}y^2z$ is compound. The simple quantities which form a compound one are called its *terms*. Thus, the terms of the foregoing compound quantity are 5, a, $-3bx$, and $\frac{1}{2}y^2z$. It is plain that the value of a compound quantity is not affected by a change in the order of its terms, provided every term retain its proper sign. Thus, $6+4$ is the same as $4+6$, and $a+b$ the same as $b+a$; the value of the compound quantity in each instance being equal to all the units contained in the two simple quantities composing it. So likewise, $b-a+d-c$ may be written $-a+b-c+d$, or $b+d-a-c$, &c.; the meaning being, that in whatever order the terms are taken, b units and d units are to be added together, and that from the sum a units and c units, or $a+c$ units, are to be taken.

13. If a compound quantity consist of two terms, such as $x+y$ or $x-y$, it is called a *binomial*; if of three, such as $x+y+z$, $x-y+z$, &c., a *trinomial*; and if of four terms or more, a *polynomial*. In comparison of such quantities, a simple quantity, such as a, or $3a^2b^3$, is termed a *mononomial*, or, as the word is usually contracted, a *monomial*.

14. When any operation is indicated as being performed on a

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* Learners should be careful to avoid the common mistake of calling *indices powers*.

compound quantity, that quantity is placed between the marks ( ), which, when thus used, are called a *vinculum*. Thus, $a(x+y)$ denotes that the compound quantity $x+y$ is multiplied by a; while, without the vinculum, $ax+y$ would mean that x alone is multiplied by a, and that y is added to the product.

When one vinculum is to be enclosed within another, the exterior one is expressed, for the sake of distinction, by the marks { }, or sometimes by [ ]. Thus, the expression, $\{a+\sqrt{(a^2+b^2)}\}^2$, means, that the square root of a^2+b^2 is added to a, and that the result is raised to the second power. *

15. Quantities which are all positive, or all negative, are said to have *like signs;* but, if some be positive and others negative, they are said to have *unlike* signs.

16. *Like quantities* are those which are expressed by the same letters, similarly combined; others are *unlike*. Thus, a, $4a$, $-3a$, are like quantities. So, also, are $5a^2b$, $-3a^2b$, and $4a^2b$; each set of quantities differing only in signs or coefficients. On the contrary, a^2b and ab^2 are unlike, as (§ 11.) they may be written aab and abb; in which expressions, though the letters are the same, they are differently combined.

17. An *equation* is an algebraic expression, which, by means of the sign $=$, denotes that two quantities are equal. Thus, $2x+5=25$, $ax^2+bx=c$, and $x^3-ax+b=0$, are equations. The quantity before the sign $=$ is called the *first* or *left-hand member* of the equation, and the one after it, the *second* or *right-hand member.*

18. An *identical equation* is one which is universally true, whatever values may be assigned to the quantities contained in it, its members differing only in form. Such, we shall hereafter find, are

$$(x+a)^2=x^2+2ax+a^2, \text{ and } (x-a)^2=(x+a)^2-4ax.$$

19. An equation is said to be of the *first, second, third,* or *nth degree* in relation to one of the quantities contained in it, and which is generally regarded as unknown, according as the highest power of that quantity is the first, second, third, or nth power. Thus, with respect to x, $ax+b=c$ is an equation of the first degree; $ax^2+bx-c=0$, one of the second; $ax^4+bx^3+cx^2=d$, one of the fourth; and $ax^n+bx=c$, one of the nth degree. An equation of

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* Instead of the characters here pointed out, the older writers used lines drawn over the quantities. In this way, the expression in the text would take the form, $\overline{a+\sqrt{\overline{a^2+b^2}}}^{\,2}$. This clumsy and inelegant form of the vinculum is now little used.

the second degree is often called a *quadratic equation*, or a *quadratic*; and one of the third degree, a *cubic equation*; while an equation of the fourth degree is sometimes called a *biquadratic equation*.

20. If an equation contain but one power of the unknown quantity, it is said to be a *simple equation*; but if it contain more than one, it is *compound*. Thus, in reference to x, $ax+b=cx$, $ax^2+b=cx^2$, $ax^n-b=3$ are simple equations; while $ax^2+bx=c$, $ax^2+bx+c=0$, and $x^6+ax^5=b$ are compound equations.

21. As it is of the utmost importance, that the student should have an accurate knowledge of the meaning of the algebraic characters and signs, so far as they have been explained, it may be proper to give some instances of the numerical computation of algebraic expressions. Thus, suppose, that at the commencement of an investigation, 10 had been denoted by a, 6 by b, and 5 by c, and that at the conclusion we had found $x=\frac{1}{2}ac+\frac{1}{3}b^2-bc+3b$; by substituting in this expression the foregoing values of a, b, and c, we get $x=\frac{1}{2}\times10\times5+\frac{1}{3}\times6^2-6\times5+3\times6$; or, by performing the operations, $x=25+12-30+18$. Hence, by adding together 25, 12, and 18, and taking 30 from the sum, we get $x=25$.

Suppose again, that in another question, a, b, c, and d had been put respectively to represent 5, 3, 8, and 7, and that we had found

$$x=\frac{a^2-b^2}{c-d}-\frac{4ab-cd}{2b-a}+\frac{12c}{a+b}-2d;$$

then, by substituting for a, b, c, and d, their values, we get

$$x=\frac{25-9}{8-7}-\frac{4\times5\times3-8\times7}{2\times3-5}+\frac{12\times8}{5+3}-2\times7=\frac{16}{1}-\frac{4}{1}+\frac{96}{8}-14$$

$=16-4+12-14$; or, by contraction, $x=10$.

For practice, the learner may compute the values of the following expressions, taking $a=5$, $b=4$, $c=2$, and $d=1$.

1. $x=\dfrac{a^3}{b+d}-\dfrac{c^3}{d}+\dfrac{b^2+3c^2}{b+c+d}-2a+3d$. *Ans.* $x=14$.
2. $x=\dfrac{a^2+b^2-d^2}{a+b+d}+\dfrac{abcd}{2b+c}-\dfrac{4a^2-10bc}{2c+d}$. *Ans.* $x=4$.
3. $y=\dfrac{3bcd+2ab^2+16}{a+b-c+d}-\dfrac{a^3-b^3-c^3+7d^3}{4bc}$. *Ans.* $y=23\frac{1}{8}$.
4. $y=\sqrt{}(a^2+b^2-5)-3\sqrt{}(a^2-b^2-c^2-d^2)$. *Ans.* $y=0$.

22. The following examples and exercises are given as a conclusion to this section, with the view of introducing the student to the mode of applying algebra in the solution of problems, and of

making him acquainted with the principal elementary processes in the resolution of equations.

Exam. 1. To find a number such that if 3 be added to its double, the sum shall be equal to what remains when the number itself is taken from 21.

To solve this question, let the required number be represented by x; then (§ 7.) its double is $2x$; to which if 3 be added, the sum (§ 6.) is $2x+3$. Again, by taking the required number x from 21, we get (§ 6.) $21-x$. Now, by the question, these two results are equal; that is, (§ 5.) $2x+3=21-x$; which (§§ 17. and 20.) is a simple equation: and this equation is plainly a *translation*, so to speak, of the conditions of the question into the brief and concentrated language of algebra. It now remains that we find the value of the unknown quantity x from the foregoing equation: and this is effected by means of certain simple and obvious processes. Thus, let x be added to both members; then, because the second member is less than 21 by x, we shall have, by § 6., the new equation, $x+2x+3=21$, or, by an obvious contraction, $3x+3=21$. From each member of this take 3; then (§ 6.) we get the equation, $3x=21-3$; or, by contraction, $3x=18$. Divide these by 3; then $x=6$; which, therefore, is the required number.* This answers the conditions of the question;

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* In mathematical operations, it is often of consequence to employ particular forms or types, suited to the nature of the subjects of inquiry. This is peculiarly advantageous when many operations are to be conducted according to one general rule or principle; such as in the computation of lunar observations for finding the longitude, in the resolution of the higher equations, and in many other instances. This will be exemplified in the form given in the margin for the resolution of the equation marked (1). In this operation (and the same will be done in many other cases), the several lines, or steps, are distinguished by the numbers 1, 2, 3, &c., enclosed in parenthetic marks, for the sake of easy reference to them in illustrating the process. In the reduction of the present equation, (2) is derived from (1) by transposition, and (3) from (2) by contraction. Equation (4) is then obtained by clearing (3) of fractions; (5) from (4) by contraction; (6) from (5) by transposition and contraction; and (7) from (6) by division. It is scarcely necessary to remark, that the numbers (1), (2), &c., need not be employed in algebraic operations, unless when the operations are to be illustrated by comments.

$$
\begin{aligned}
&3x-\frac{2x}{7}=2x+\frac{x+1}{2}+1 \ldots\ldots (1)\\
&3x-2x-\frac{2x}{7}=\frac{x+1}{2}+1 \ldots\ldots (2)\\
&x-\frac{2x}{7}=\frac{x+1}{2}+1 \ldots\ldots\ldots\ldots (3)\\
&14x-4x=7x+7+14 \ldots\ldots (4)\\
&10x=7x+21 \ldots\ldots\ldots\ldots\ldots\ldots (5)\\
&3x=21 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)\\
&x=7 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)
\end{aligned}
$$

since if 3 be added to the double of 6, the sum is 15; and if 6 be taken from 21, the remainder is also 15.

Exam. 2. Required a number, such, that if it be multiplied by 7, and 1 be taken from the product, one third of the remainder shall be greater by 17 than the number itself.

Here, denoting the required number by x, we get, successively, by performing the operations mentioned in the question, by means of §§ 7, 6, and 8., $7x$, $7x-1$, and $\frac{7x-1}{3}$; which last is to be equal to x increased by 17, that is, to $x+17$. We have, therefore, for the algebraic expression of the question, the equation, $\frac{7x-1}{3}=x+17$. To find the value of x from this equation, let both its members be multiplied by 3: then, since 3 times the third part of $7x-1$ is evidently $7x-1$, we get $7x-1=3x+51$. By adding 1 to both members of this, we obtain $7x=3x+51+1$, or $7x=3x+52$. From these equals take $3x$: then $7x-3x=52$ or $4x=52$. Lastly, since the fourth part of $4x$ is evidently x, we get, by dividing these equals by 4, $x=13$, the number required. To try the correctness of this result, we multiply, according to the question, 13 by 7, and from the product, 91, we take 1; then, dividing the remainder by 3, we get 30, which is equal to 13 and 17, or is greater than 13 by 17.

23. From a review of the operations in the two preceding questions, it will be seen that we have employed the following axioms:—

Axiom 1. If equals, or the same be added to equals, the wholes or sums are equal.

Ax. 2. If equals, or the same be taken from equals, the remainders are equal.

Ax. 3. If equals be multiplied by the same, or by equals, the products are equal.

Ax. 4. If equals be divided by the same, or by equals, the quotients are equal.

24. To *resolve an equation*, that is, to find the value of some assigned quantity contained in it, and regarded as unknown, it is necessary to separate the required quantity and the others. This, as we have seen, is effected in the foregoing examples by means of the axioms in the last §; and the same can be done in every like case, in a similar manner. From these principles, however, technical rules may be derived, which will be found to be more convenient for practical use. Thus, from the equation $x+p=q$,

we derive another equation, $x=q-p$, by subtracting p from both members; while, had the original equation been $x-p=q$, we should have got $x=q+p$, by adding p to both its members. It appears, therefore, that whether the sign of any term, such as p, be $+$ or $-$, another equation will be obtained by removing that term to the other member and changing its sign. Hence,

25. *A quantity may be* transposed *from one member of an equation to the other, by changing its sign.* It is plain, also, that *any* number of terms may be transposed in a similar way at a single operation. Thus, if $4x-a+b=3x+c$, we shall have $4x-3x=a-b+c$, or $x=a-b+c$.

26. Again, if $3x=12$, we get the new equation $x=4$, by dividing both members by 3; and it is plain that a like division might be made, if, instead of $3x$, we had $5x$, $10x$, or, in general, ax. Hence, *in an equation a* multiplier *of any term may be removed by* dividing *both members by it.*

27. If $\frac{x}{5}=4$, we get the new equation $x=20$, by multiplying each member by 5; and if, instead of 5, the divisor were 3, 7, or any other number, such as a, the divisor would be removed by multiplying by 3, 7, or a. We have, therefore, the following rule: — *In any equation a* divisor *or* denominator *may be removed by* multiplying *both members by it.* This process is generally called *the clearing* or *the freeing of an equation of fractions.*

28. *If there be two or more fractions in an equation, they may all be removed at a single operation* (1.) *by multiplying by them all at once (or by their product), or* (2.) *by their least common multiple.* The first of these two processes is effected most simply by multiplying each numerator by all the denominators except its own, and then omitting its own, and by multiplying all the quantities which are not fractions by all the denominators successively, or by their product. Thus, if we have $\frac{2x}{3}-\frac{x}{4}+\frac{2x}{5}=49$, we multiply $2x$ by 4 and the product by 5, or $2x$ at once by $20(=4\times5)$; $-x$ by 3 and 5, or by 15; $2x$ by 3 and 4, or by 12; and 49 by 3, 4, and 5, or by $60(=3\times4\times5)$. We thus get $40x-15x+24x=2940$, or $49x=2940$; whence (§ 26.) $x=60$. It is plain, that in this process each term is multiplied by $3\times4\times5$, or by 60: and therefore all the parts of the first member being increased sixty-fold, and the second member being increased in the same degree, the two resulting members (Ax. 3. p. 9.) must evidently be equal.

It may be remarked, that, for completing the elementary theory of the reduction of equations, it would be necessary to give the method of freeing equations from radical quantities, when they occur. This, however, will be given with more advantage in a subsequent section.

*Exam.** 3. Find the value of x in the equation

$$6x-5=15-4x.$$

Here, by transposition (§ 24.), $6x+4x=15+5$, or $10x=20$. Hence, by dividing by 10 (§ 26.) we get $x=2$.

To prove the correctness of the result, if we substitute 2 for x in the given equation, we get $6\times 2-5$, or $12-5$, that is 7, for the first member; and for the second, $15-4\times 2=15-8=7$. Hence the value found for x is correct, as it makes the two members of the given equation equal. The student will find it useful to prove his operations in the manner that has now been pointed out.

Exam. 4. Resolve the equation $x-7=\frac{x-1}{4}$.

By clearing this of fractions (§ 27.) we get $4x-28=x-1$; whence, by transposition (§ 24.), $4x-x=28-1$, or $3x=27$. Dividing, therefore, by 3 we get $x=9$, the answer.

Exam. 5. Resolve the equation $\frac{x+1}{6}+\frac{x-1}{5}=4$.

Here (§ 27.) by multiplying $x+1$ by 5, and omitting the denominator, we get $5x+5$; by multiplying $x-1$ by 6, and omitting 5, we find $6x-6$; and, by multiplying 4 by 6 and the product by 5, or 4 at once by 30, we obtain 120. Hence the equation is changed into $5x+5+6x-6=120$. Hence, by transposition, $5x+6x=120-5+6$, or, by contraction, $11x=121$; whence, by dividing by 11, we get $x=11$, the answer.

When there are fractions it is generally best to commence by removing them, as was done in the last example. Sometimes, however, especially in the more complicated expressions, it is preferable first to transpose certain terms, as by this means contractions may often be made. The following equation exemplifies this.

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* As it is of great consequence for the student to acquire facility and expertness in the performance of algebraic operations, some numerical examples are here given; and a number of exercises are subjoined, which can all be wrought by the principles that have been established. To the examples he ought to attend, till he can work them without assistance from the book or from his teacher; and he will then find it easy to work the exercises.

Exam. 6. Resolve the equation $3x-\frac{x}{2}+5=2x+8$.

From this we get, by transposition, $3x-2x-\frac{x}{2}=8-5$; or, by contraction, $x-\frac{x}{2}=3$. Then, by multiplying by 2, to clear the equation of fractions, we get $2x-x=6$, or $x=6$, the answer.

Exam. 7. Find the value of y in the equation $\frac{5y}{6}-6=\frac{7y}{12}$.

Here, since 12 is a multiple of 6, instead of multiplying by 6 and 12, we multiply only by the latter. In this way, the first term becomes $\frac{60y}{6}$, or $10y$; and the equation is changed into $10y-72=7y$; whence, by transposition, &c., we get $y=24$. In multiplying the first term by 12, we might have multiplied by 6, which, by removing the denominator 6, would give $5y$; the product of which by 2, the remaining factor of 12, gives $10y$, as before. A like process is admissible in every similar case.

Exam. 8. Resolve the equation $\frac{x}{4}+\frac{5x}{6}-\frac{7x}{9}=22$.

Here, by multiplying by 4, the product by 6, and that product by 9, we get, successively, $x+\frac{20x}{6}-\frac{28x}{9}=88$;

$$6x+20x-\frac{168x}{9}=528, \text{ or } 26x-\frac{168x}{9}=528;$$

$$\text{and } 234x-168x=4752, \text{ or } 66x=4752;$$

the division of which by 66 gives $x=72$, the answer.

To solve this according to the second method mentioned in § 28., we have (*Arith.* p. 83.) 36, the least common multiple of 4, 6 and 9. Multiplying, therefore, the first fraction by 36 $(=4\times9)$, we get $9x$; multiplying the second by 36 $(=6\times6)$, we find $30x$; and multiplying the third by the same $(=9\times4)$, we obtain $-28x$. Multiplying also 22 by 36 we get 792, and the equation is changed into $9x+30x-28x=792$, or $11x=792$; whence, by dividing by 11, we get $x=72$, as before. This method has the advantage of keeping the numbers smaller and more manageable.

Exam. 9. Find two numbers differing by 2, and such that the greater and four times the less may be together equal to 27.

Let x be put to denote the less; then, by the question, $x+2$

will be the greater. By the question, also, $x+2+4x$, or $5x+2=27$; whence, by transposing 2 and dividing by 5, we get $x=5$, the less number; and therefore the greater is 7. We might here have assumed x to denote the greater number, and then $x-2$ would represent the less, and the work would proceed in almost the same manner as before.

Exam. 10. A farmer rents 100 acres of land for 150*l.*, part at 27*s.* and part at 37*s.* per acre. How many acres has he of each kind?

Let x denote the number of acres of the dearer kind, and $100-x$ will be the acres of the other. To find the rents of these in shillings, multiply the former by 37, and the latter by 27; the results are $37x$ and $2700-27x$. Now, by the question, the sum of these rents must be 3000, the shillings in 150*l.* We have, therefore, $37x+2700-27x=3000$; whence by transposing 2700 and contracting, we get $10x=300$; and, by division, $x=30$, the number of acres of the dearer kind; by taking which from 100, we get 70 acres, the quantity of the cheaper kind.

Exam. 11. Divide 51 into four parts, such that one half of the first, one third of the second, one fifth of the third, and one seventh of the fourth may be all equal.

Let x denote each of the equals, one half of the first, one third of the second, &c. Then will the required parts be $2x$, $3x$, $5x$, and $7x$. But, by the question, the sum of these is to be 51; that is, $2x+3x+5x+7x$, or $17x=51$; whence, $x=3$: and by multiplying this successively by 2, 3, 5, and 7, we get 6, 9, 15, and 21, the required numbers.

Exam. 12. If one person, A, start from a certain place, and travel at the rate of 9 miles an hour; and if, two hours and twenty minutes after, another person, B, start in pursuit of him, and travel at the rate of 11 miles an hour; how far will each have travelled before A is overtaken?

To solve this question, let x be the number of hours that B travels before overtaking A. Then, in that time B will travel $11x$ miles, and A $9x$ miles. Before B started, however, A had travelled two hours and a third at the rate of 9 miles per hour; he must, therefore, have travelled twice 9 and the third of 9, or 21 miles. Hence, by the nature of the question, $11x-9x$, or $2x=21$; whence, by dividing by 2, $x=10\frac{1}{2}$ hours, the time travelled by B, and the product of this by 11 is $115\frac{1}{2}$ miles, the distance travelled by each. By adding 2 hours 20 minutes to 10 hours 30 minutes, we get 12 hours 50 minutes, or $12\frac{5}{6}$ hours, the time A travelled; the product of which by 9 is $115\frac{1}{2}$, the space travelled; and this proves the correctness of the result. In this solution, instead of x being put to represent the required distance, it has been put to denote the time that B travelled; and by this means the solution has been considerably facilitated. This is one instance, out of many, in which it is better to put x to represent, not the quantity ultimately required, but one from which that quantity may be derived.

Exam. 13. If a person each year double his capital except an expenditure of 400*l.*, and at the end of three years find himself worth three times his original capital, what had he to commence with?

Let x be his original capital in pounds: then at the end of the first year, he will have $2x-400$, which is his capital at the beginning of the second. From the double of this, $4x-800$, take 400, and the remainder, $4x-800-400$, or $4x-1200$, will be his capital at the beginning of the third year. Double this in like manner and subtract 400; and the result $8x-2400-400$, or $8x-2800$, will be the amount of his property at the end of the third year. Then, by the question, $8x-2800=3x$; whence, by transposition, &c. $x=560l.$, the answer.

Proof. $560\times2=1120$, and $1120-400=720$, his property at the end of the first year. $720\times2=1440$, and $1440-400=1040$, his capital at the end of the second year. Lastly, $1040\times2=2080$, and $2080-400=1680$, what he was worth at the end of the third year; and as this is exactly treble of the original capital, 560*l.*, the answer is correct.

Exercises in Numerical Equations of the First Degree.

Resolve the following equations:—

1. $6x+31=281-4x$. *Ans.* $x=25$.
2. $11+x=319-3x$. *Ans.* $x=77$.
3. $2x-9=72+\frac{x}{5}$. *Ans.* $x=45$.
4. $x-11=\frac{x+2}{5}+7$. *Ans.* $x=23$.
5. $\frac{x}{2}-1=\frac{x}{3}+1$. *Ans.* $x=12$.
6. $11-\frac{x}{5}=13-\frac{x}{4}$. *Ans.* $x=40$.

7. $\frac{x+1}{4}+\frac{x-1}{6}=8.$ *Ans.* $x=19.$

8. $\frac{x-3}{8}+\frac{x+9}{12}=\frac{3x+7}{20}+3.$ *Ans.* $x=51.$

9. $\frac{2x}{3}+\frac{4x}{5}-\frac{6x}{7}=\frac{x}{2}+\frac{3x}{4}-\frac{5x}{6}+81.$ *Ans.* $x=420.$

10. $\frac{x-1}{2}+\frac{x-2}{3}-\frac{x-3}{4}=6.$ *Ans.* $x=11.$

11. Divide 11 into two parts, such that the sum of twice the first and half the second may be 16. *Ans.* 7 and 4.

12. Divide 39 into four parts, such, that if the first be increased by 1, the second diminished by 2, the third multiplied by 3, and the fourth divided by 4, the results may be all equal. *Ans.* 5, 8, 2, 24.

13. Suppose two coaches to start at the same hour, one from London for Glasgow, and the other from Glasgow for London, the former travelling $10\frac{1}{2}$ and the latter $9\frac{1}{2}$ miles per hour: where will they meet, the distance between the two cities being 400 miles? *Ans.* 210 miles from London.

14. Suppose every thing to be as in the last question, except that the coach from Glasgow starts two hours earlier than the other; where will they meet? *Ans.* $200\frac{1}{40}$ miles from London.

15. A dealer purchases 60 yards of cloth for 30*l.*; and by selling one part of it at 12*s.*, another, twice as great, at 14*s.*, and the rest at 10*s.* per yard, he gains 8*l.* How many yards were in the several lots? *Ans.* 16, 32, and 12.

16. Suppose two dealers each annually to double his capital except an expenditure of 100*l.*; and, that at the end of three years, the capital of the one is found to be doubled, while the other has only half what he had at first: how much had each to commence with? *Ans.* 116*l.* 13*s.* 4*d.* and 93*l.* 6*s.* 8*d.*

17. If a person each year double his capital except an expenditure of 300*l.* the first year, 400*l.* the next year, and 500*l.* the third, and at the end of three years be found to be worth 5500*l.*, what was his original capital? *Ans.* 1000*l.*

18. A father's age is now treble of his son's, while five years ago it was quadruple: what are their present ages? *Ans.* 45 and 15 years.

19. Divide 1000*l.* between A, B, and C, giving A 100*l.* more,

and B 50*l.* less, than C. *Ans.* A's share 416*l* 13*s.* 4*d.*; B's 266*l.* 13*s.* 4*d.*; and C's 316*l.* 13*s.* 4*d.*

20. A spirit merchant finds that if he add 10 gallons of water to a cask of brandy, the mixture will be worth 21*s.* per gallon; but that if he add 10 gallons more, the value will be reduced to 18*s.* How many gallons were in the cask? *Ans.* 50.

21. Find a number, such that if it be divided successively by 2, 3, 4, 5, 6, 7, 8, 9, and 10, half the sum of the first four quotients increased by 20 shall be equal to the sum of the remaining five. *Ans.* 5040.

22. Find two numbers differing by 6, and such that three times the less may exceed twice the greater by 7. *Ans.* 25 and 19.

23. Find a number such, that if it be increased successively by 1, 2, and 3, the sum of one half of the first result and one third of the second shall exceed one fourth of the third by 8. *Ans.* 13.

CHAPTER II.

FUNDAMENTAL OPERATIONS.

ADDITION.

29. *To add together unlike quantities;* write them in succession, prefixing to each its proper sign.

Thus, (§ 6.) the sum of a and b is $a+b$; while that of $x-y$ and $v-z$ is $x-y+v-z$, the sign + being omitted in the one sum before the first term a, and in the other before the first term x.

Like quantities might also be added in a similar manner. Thus, the sum of a and a might be expressed by $a+a$, that of $2a$ and $3a$ by $2a+3a$, &c. Such quantities, however, may be *incorporated,* that is, may have two or more terms combined into one, by means of the two following rules.

30. *To add quantities which are like, and have like signs;* add the coefficients together, and to their sum prefix the common sign, and annex the common quantity.

Thus, the sum of $3a$ and $5a$ is $8a$; and that of $2a-3b$ and $6a-4b$ is $8a-7b$. The reason is plain. The sum of 3 times any quantity a and 5 times the same will be 8 times that quantity. Thus, the sum of 3 days and 5 days is 8 days, and 3 shillings and 5 shillings are together equal to 8 shillings. In the second example, the sum of $2a$ and $6a$ is $8a$: but the first quantity is less than $2a$ by $3b$, and the second is less than $6a$ by $4b$. The sum $8a$ must, therefore, be diminished by the sum of $3b$ and $4b$, that is $7b$, to find the correct sum of $2a-3b$ and $6a-4b$.

If the coefficients be literal, they may be added according to § 29., and the common quantity annexed to their sum placed in a vinculum. Thus, the sum of ax, bx, and cx, may be written either

$$ax+bx+cx, \text{ or } (a+b+c)x.$$

In the two following examples, the like quantities are placed in the same column; an arrangement which facilitates the process, and lessens the chance of falling into mistakes; and which ought, therefore, to be adopted in other cases, as far as it conveniently can.*

Exam. 1.

$$\begin{array}{rr}
 & 3a^2x^3-4ax^2-3x \\
 & 5a^2x^3-2ax^2-x \\
 & 4a^2x^3-ax^2-5x \\
\cline{2-2}
\text{Sum} = & 12a^2x^3-7ax^2-9x
\end{array}$$

Exam. 2.

$$\begin{array}{rr}
 & 1-2x+3x^2-4x^3+5x^4 \\
 & 2-3x+4x^2-5x^3+6x^4 \\
 & 3-4x+5x^2-6x^3+7x^4 \\
\cline{2-2}
\text{Sum} = & 6-9x+12x^2-15x^3+18x^4
\end{array}$$

31. *To add quantities which are like, but which have unlike signs;* add together first the positive, and then the negative coefficients; subtract the less sum from the greater; to the remainder prefix the sign of the greater; and annex the common quantity.

Exam. 3.

$$\begin{array}{r}
4x^3-5ax^2-a^2x+3a^3 \\
-3x^3+ax^2-3a^2x+5a^3 \\
2x^3+6ax^2+5a^2x-2a^3 \\
x^3-3ax^2-4a^2x-3a^3 \\
\hline
4x^3-ax^2-3a^2x+3a^3
\end{array}$$

Exam. 4.

$$\begin{array}{r}
3x^5y-4x^3y^3+xy^5 \\
x^5y+3x^3y^3+3xy^5 \\
-3x^5y+x^3y^3-6xy^5 \\
\hline
x^5y\qquad\qquad-2xy^5 \\
\text{or } x^5y-2xy^5
\end{array}$$

Exam. 5.

$$\begin{array}{r}
-a+b+c \\
a-b+c \\
a+b-c \\
\hline
a+b+c
\end{array}$$

In the first column of the first of these examples, the sum of the positive coefficients, 1, 2, and 4, is 7; from which we take 3, the only negative coefficient: the remainder 4 is positive, and therefore the sum of the quantities in the first column is $4x^3$, to which, as it is the first term, it is unnecessary to prefix the sign +. In the second column, the sum of the positive coefficients is 7, and that of the negative 8. Taking the former from the latter, we get 1, to which the sign —, that of the greater, is to be prefixed; and, therefore, the sum of the quantities in the column is $-1ax^2$, or as it is written, $-ax^2$. The other columns are added in a similar manner.

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* In common arithmetic we commence operations in addition, subtraction, and multiplication, at the right-hand side, and proceed towards the left, on account of "carrying" from column to column. In algebra no advantage would be gained by following that order; and therefore, in these rules, we commence at the left side, and proceed in the order in which quantities are written and read.

The reason of the process is plain. In the first column, the positive terms amount to $7x^3$, which is to be diminished by the subtractive term $-3x^3$. In the second column, the amounts of the positive and negative terms are respectively $7ax^2$ and $-8ax^2$. What is to be subtracted, therefore, exceeds what is to be added by once ax^2; and this is accordingly written down, with the sign of subtraction prefixed.

In Exam. 4. the amounts of the positive and negative coefficients in the second column are equal. Hence, the positive and negative terms neutralise each other, as much being subtracted as there is added; and, therefore, the sum of the quantities in that column is nothing.

32. In a great proportion of cases in actual practice, all the three foregoing rules, or two of them at least, must be employed. Thus, in the annexed example, the first column is added according to § 30., the second by § 31., and the third by § 29.

Exam. 6.

$$\begin{array}{l} 2a^2x-3ax+4 \\ 5a^2x+4ax-b \\ \;\;a^2x-2ax+2c \\ \hline 8a^2x-ax-b+2c+4 \end{array}$$

Exercises.[*] —Required the sums of the following quantities: —

1. ax^2, a^2x, y^3, and $3by^2$. *Ans.* $ax^2+a^2x+y^3+3by^2$.

2. $a^3-2a^2b-3ab^2+2b^3$, $3a^3-a^2b-4ab^2+10b^3$, $2a^3-3a^2b-6ab^2+b^3$, and $5a^3-4a^2b-ab^2+3b^3$.
Ans. $11a^3-10a^2b-14ab^2+16b^3$.

3. x^3-5x^2-3x+1, $2x^3+6x^2+5x+3$, $3x^3-2x^2-x-1$, and $4x^3-x^2+2x-5$. *Ans.* $10x^3-2x^2+3x-2$.

4. $-a+b+c+d$, $a-b+c+d$, $a+b-c+d$, and $a+b+c-d$.
Ans. $2a+2b+2c+2d$.

5. $a-2b$, $2b-3c$, $3c-4d$, $4d-5e$, and $5e-6f$. *Ans.* $a-6f$.

6. $y^3+2y^2z-3yz^2$, $2y^3+2y^2z+5yz^2$, and $3y^3-4y^2z-2yz^2$.
Ans. $6y^3$.

7. ax^3+bx^2, bx^3-cx^2, and cx^3+dx^2.
Ans. $(a+b+c)x^3+(b-c+d)x^2$.

8. mz^2-nz, nz^2-pz, and $2z^2-z$.
Ans. $(m+n+2)z^2-(n+p+1)z$.

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[*] The number of exercises given here, as well as in subtraction, is purposely small; as ample practice, and of the best kind, is supplied in multiplication and division.

SUBTRACTION.

33. *To subtract one quantity from another;* conceive the sign or signs of the former to be changed (+ into −, and − into +), and proceed as in addition.

To prove this important rule, by means of which, without any diversity of cases, we are enabled to perform the subtraction of quantities by the rules of addition already given, let us assume two quantities, $x-y$ and $z-v$, each having one term positive and the other negative, and thus presenting every variety of signs; and let it be required to subtract the latter from the former. Now, if from $x-y$ we take z, the remainder (§ 6.) is $x-y-z$. The quantity to be subtracted, however, is not z but $z-v$, that is, a quantity less than z by v. Having thus, therefore, taken away too much by v, we have left too little by the same quantity, and we must accordingly add v. Hence (§ 6.) the true result is $x-y-z+v$. Comparing this with the original quantities, $x-y$ and $z-v$, we see that the signs of $x-y$ remain the same as they were, but that the signs of $z-v$, the quantity to be subtracted, are changed, the first from + to −, and the second from − to +: and, finally, we see, that after these changes are made, the quantities are connected as in addition.*

$$\begin{array}{r} x-y \\ z-v \\ \hline x-y-z+v \end{array}$$

Had the given expressions consisted of like quantities, instead of being unlike, as those above employed, every thing would have proceeded on the same principle; but in such a case, the like quantities would admit of incorporation according to § 30. or § 31. Thus, in the annexed example, by proceeding as before, we should get $5a-2x-2a+3x$, or, by contraction, $3a+x$, the same that would be obtained by changing $2a$ into $-2a$, and $-3x$ into $3x$, and then adding the columns according to § 31.

$$\begin{array}{r} 5a-2x \\ 2a-3x \\ \hline 3a+x \end{array}$$

In case of like quantities, having like signs, when the coefficient of the one to be subtracted is the less, we may simply subtract the less coefficient from the greater, and to the remainder prefix the common sign, and annex the common quantity. Thus, in the last example, by taking $2a$ from $5a$ we get $3a$. So also, if $-7x$ be taken from $-12x$, the remainder is $-5x$. In other cases the general rule is to be employed. When there are literal coefficients, let the operation be performed on them, and to the remainder annex the common quantity, employing the vinculum, if necessary. Thus, if by be taken from $(a+b)y$, the remainder is ay; and if nx be taken from mx, the remainder is $mx-nx$, or $(m-n)x$. So likewise, if from az we take $(a-b)z$, we get bz.

---

* Since $a-a=0$, and $-a+a=0$, and since in subtraction the remainder must be such, that if to it the quantity subtracted be added, the result must be the other one of the proposed quantities; the rule for subtraction may be easily established on this principle. Thus, if from x we are to subtract $y-z$, the remainder must be $x-y+z$; since by adding $y-z$ to this we get $x-y+z+y-z$, or, by contraction, x: and hence we see that the signs of y and $-z$ must be changed.

Exam. 1.	*Exam.* 2.	*Exam.* 3.
$6x^4z^2-a^6$	$x^4+4x^3y+6x^2y^2+4xy^3+y^4$	$4a^3-8a^4+16a^5$
$2x^4z^2-b^6$	$x^4-4x^3y+6x^2y^2-4xy^3+y^4$	$3a^3-4a^4+5a^5$
$4x^4z^2-a^6+b^6$	$8x^3y \qquad +8xy^3$	$a^3-4a^4+11a^5$

Exercises.

1. From $a+b+c$, take $-a+b+c$. *Ans.* $2a$.
2. From $4a^3-3a^2x+ax^2-2x^3$, take $2a^3-4a^2x-ax^2-x^3$. *Ans.* $2a^3+a^2x+2ax^2-x^3$.
3. Take x^3-3x^2+3x-1 from x^3+3x^2+3x+1. *Ans.* $6x^2+2$.
4. From ax^2+by^2 take cx^2-dy^2. *Ans.* $(a-c)x^2+(b+d)y^2$.

34. We have thus far considered negative quantities only in connexion with positive ones in compound quantities. Such quantities, however, often occur in a detached form, and it may be proper here to enter into some consideration of their nature and character.

If from a we take b, the expression for the remainder is $a-b$; and so long as a is not less than b, the meaning is plain. Thus, if $a=5$ and $b=2$, we have $a-b=5-2=3$. In this case we may put b under the form $5-3$; and by taking this from $(a=)5$, by the rule for subtraction, we get 3 for remainder, as before. If, however, a be less than b, the subtraction, though it may be indicated to the eye, can no longer be performed in the ordinary arithmetical sense. Thus, if $a=5$ and $b=8$, we should have $a-b=5-8$, which, in the arithmetical sense, denotes an operation that cannot be performed. Putting, however, 8 under the form $5+3$, and subtracting this from 5 by the rule for subtraction, we get for remainder -3. To interpret this expression, we have to consider, that, by an extension of the rule for subtraction not originally contemplated, we have gone through the *form* of subtracting $5+3$

from 5. The first term of $5+3$ we *can* subtract from 5, and the remainder is 0; but there still remains 3 to be subtracted; and this is what is indicated by the expression -3. This expression, therefore, denotes a *defect*, and shows that 3 still remains unsubtracted, and that it must be subtracted afterwards, if there be opportunity. Thus, if $a=5$, $b=8$, and $c=7$, the expression, $a-b+c$, becomes $5-8+7$; and by taking 8 from 5, as above, we get -3, a negative or subtractive quantity. Taking this, therefore, from 7, we get 4 as the value of $a-b+c$, the same that would be obtained by taking 8 from $5+7$.*

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* As a farther illustration, if a person have an income of 1000*l.* a year, and spend 600*l.*, he makes a saving of 400*l.*; if he spend 900*l.*, his saving is reduced to 100*l.*; and if he spend 1000*l.*, it becomes nothing. If, however, he spend 1200*l.* there is no longer a *saving* in the ordinary meaning, but a *defect*, or a reduction upon his other savings, of 200*l.* This the algebraist, still keeping up the idea of a saving, regards as subtractive or negative, and in this way he still arrives at the same final result. By this means, the number of rules for the performance of operations is greatly diminished, and a uniformity of conception and process is obtained, which is of high importance.

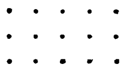
From these views it will be seen, that the adding of a negative quantity, in the algebraic sense, is the same as the subtracting of it, in the arithmetical sense. Thus, in the foregoing illustration, the savings in the first three years amount to 500*l*; and if to this we add -200*l.*, the saving in the algebraic sense, in the fourth year, we get 300*l.*, the entire amount of the savings in the four years. On the other hand, the subtraction of a negative quantity is, in the arithmetical sense, the same as the addition of that quantity taken as positive. Thus, to make use of the same illustration, if we suppose the person to have employed his savings in the gradual extinction of a debt, that is, if he paid every year 1000*l.* diminished by his expenditure, he would have reduced the debt by 400*l.* in the first year, by 100*l.* in the next, and by nothing in the third; while in the fourth he would have increased it by 200*l.*, his negative saving taken positively.

These considerations will lead us to see, that, of two negative quantities, the one which is less in *absolute* value, that is, the one which contains fewer units, is greater than the other, in comparison with positive quantities. Thus, if from 5 we take successively 1, 3, 8, and 10, we get the remainders, 4, 2, -3, and -5, each of which after the first, is evidently less than the one before it; so that -3 is greater than -5.

Hence also, according to the same views, we shall see the correctness of the somewhat startling assertion, that *a negative quantity is less than* 0. This will be plain from considering, that if to a we add 0, we have still a; while if, in the algebraic sense, we add $-b$, the result is $a-b$, which is less than a; and therefore $-b$, the quantity last added, is less than 0, which was added in the former case. In a similar sense, it is often said in common language, that if a man's debts exceed his property, he is worth less than nothing.

MULTIPLICATION.

35. *When factors are multiplied together, the product is the same in whatever order the operation is performed.* Thus, 3 times 5 is the same as 5 times 3. To illustrate this, let dots be assumed to represent objects, and let three lines, each containing five dots, be placed as in the margin. We have thus 15 objects represented, which may be regarded either as 3 times 5 objects, when we take the three horizontal rows; or as 5 times 3 objects, when we take the five vertical columns, each of which contains 3 dots; and a similar illustration may be given in every case.



In like manner, the continual product of 2, 3, and 5 will be the same, in whatever order these factors are taken. Thus, by what we have seen, it will be either 2.3.5 or 2.5.3, that is, 2×15; or, as we have seen, 15×2; or, which is still the same, 3.5.2, or 5.3.2: and other variations would be obtained by putting these last under the forms, 3×10 and 5×6.

On the same principle we should have, universally, ab the same as ba; and abc, acb, bac, bca, cab, and cba are all equivalent.*

36. *When quantities are multiplied by a positive term, their signs are retained in the product; but when by a negative one, they are changed.*

To prove this most important rule, let us multiply $x-y$ by $a-b$, each of these quantities exhibiting every variety of signs. Now, if we multiply $x-y$ by a, the product will be a times x *minus* a times y, that is $ax-ay$.† The multiplier, however, is

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* To show this otherwise, and by more general reasoning, we have a times 1 unit obviously the same as a units, or once a units; and a times 2 units will be double of this or will be $2a$ units; that is, $a \times 2 = 2 \times a$. In like manner, we should have $a \times 3 = 3 \times a$, $a \times 4 = 4 \times a$; and, in general, $a \times b = b \times a$, or $ab = ba$; so that it is the same whether we take a as the multiplicand and b as the multiplier, or the reverse. Put now p to denote ab or ba, and multiply by c: then, by what we have seen, we have $pc = cp$; whence, by restoring the values of p, we get $abc = cab = bac = cba$: and, by regarding ca and cb, in the second and fourth of these, each as a single factor, we should have bca and acb each equivalent to any of the other products. By multiplying by other factors, such as d, e, &c., the principle would be shown to hold universally.

† That the product of $x-y$ by a is $ax-ay$ may be illustrated by addition, as in the margin, where the amount of $x-y$ taken 3 times is shown to be $3x-3y$: and the same may be shown regarding $x-y$ taken twice, four times, or in general a times. It would appear in the same manner, that the product of $x+y$ by a is $ax+ay$. As a familiar illustration, if a *false* pound weight

$$\begin{array}{c} x-y \\ x-y \\ \underline{x-y} \\ 3x-3y \end{array}$$

not a, but $a-b$; a quantity less by b than a. Hence we are to multiply $x-y$ by b, and to subtract the product, $bx-by$, from the former product, $ax-ay$. To make this latter subtraction be performed by the rules for addition in §§ 29, 30, and 31., we change (§ 33.) the signs of $bx-by$, and then the work stands as in the second part of the annexed operation, the addition being performed in this example by § 29. By examining the two partial products, $ax-ay$ and $-bx+by$, in this latter part, we see, that in the first, which is produced by multiplying by the positive term a, the signs of the two terms in the product are respectively the same as those of $x-y$, the multiplicand; while in the second, which arises from multiplying by the negative term, $-b$, the signs are the opposite of those in the multiplicand, $x-y$. The same may be shown in every case; and the truth of the rule is thus established.

$$\begin{array}{l} x-y \\ a-b \\ \hline ax-ay \\ \qquad bx-by \end{array}$$

$$\begin{array}{l} x-y \\ a-b \\ \hline ax-ay \\ \qquad -bx+by \\ \hline ax-ay-bx+by \end{array}$$

This rule is often expressed briefly in the following terms: *In multiplication, like signs produce* plus *in the product, and unlike signs* minus. The mode of expression given above, however, is more simple and natural.

37. *To multiply terms which have coefficients;* find the product of the coefficients, and prefix it to the product of the other quantities.

Thus, the product of $2a$ and $3b$ is $6ab$. The reason is plain, since the product is $2\times a\times 3\times b$, or (§ 35.) $2\times 3\times a\times b$, or, which is the same, $6ab$.

38. *The product of two or more powers of the same quantity, is expressed by writing that quantity with an index equal to the sum of the indices of the proposed powers.*

Thus, the product of a^2 and a^3 is a^5; and the continual product of x^3, x^4, and x^6 is x^{13}. So likewise the product of x^m and x^n is x^{m+n}, and that of x and x^n is x^{n+1}: and, on the same principle, the product of x^{m-n} and x^n is x^m.

The reason of this is evident from §§ 10. and 7. Thus a^2 and a^3 are the same as *aa* and *aaa;* the product of which is *aaaaa* or a^5; the index 5 being the sum of the indices 2 and 3, the numbers which show how often a is used as a factor in the given powers.

be a pound wanting an ounce, two quantities weighed by it would be two pounds wanting two ounces, and a quantities weighed by it would be a pounds wanting a ounces.

*Examples.**

Exam. 1.

$$\begin{array}{r}3a\\4x\\\hline 12ax\end{array}$$

Exam. 2.

$$\begin{array}{l}4x+3y-z\\2a\\\hline 8ax+6ay-2az\end{array}$$

Exam. 3.

$$\begin{array}{l}a^3-5a^4+2a^6\\3a^2\\\hline 3a^5-15a^6+6a^8\end{array}$$

Exam. 4.

$$\begin{array}{l}z^{m-n}-2z^m+3z^{m+n}\\3z^n\\\hline 3z^m-6z^{m+n}+9z^{m+2n}\end{array}$$

Exam. 5.

$$\begin{array}{lllll}x^3 & -2ax^2 & +4a^2x & & \\ 3x^2 & -ax & -2a^2 & & \\\hline 3x^5 & -6ax^4 & +12a^2x^3 & & \\ & -ax^4 & +2a^2x^3 & -4a^3x^2 & \\ & & -2a^2x^3 & +4a^3x^2 & -8a^4x\\\hline 3x^5 & -7ax^4 & +12a^2x^3 & & -8a^4x\end{array}$$

Exam. 6.

$$\begin{array}{llllll}a^3 & -3a^2b & +b^3 & & & \\ 2a^2 & -2ab & -3b^2 & & & \\\hline 2a^5 & -6a^4b & & +2a^2b^3 & & \\ & -2a^4b & +6a^3b^2 & & -2ab^4 & \\ & & -3a^3b^2 & +9a^2b^3 & & -3b^5\\\hline 2a^5 & -8a^4b & +3a^3b^2 & +11a^2b^3 & -2ab^4 & -3b^5\end{array}$$

In the first and second of these examples, the work is performed by means of § 37.; and, in addition to this §, the 3d, 4th, 5th, and 6th examples require the use of § 38. In all of them, likewise, the principle established in § 36. is employed. In Exam. 6. we incorporate $2a^2b^3$ in the third column with $9a^2b^3$ in the fourth, these terms being like.

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* In the more lengthened algebraic operations in multiplication, the following very simple and easy verification of the process, so far as the coefficients are concerned, will be found useful. Find the separate sums of the coefficients of the multiplicand, multiplier, and product: then, if the last of these be unequal to the product of the other two, the work must be wrong; but, if it be equal to their product, the work may be presumed to be correct. Thus, in Exam. 6., the sum of the coefficients of the multiplicand, multiplier, and product are respectively —1, —3, and 3; the last of which is the product of the other two; and therefore the coefficients may be supposed to be correct. Should the complete product be found to be wrong, the partial products might be tested in a similar manner. Thus, in the same example, by multiplying —1 successively by 2, —2, and —3, the coefficients of the multiplier, we get —2, 2, and 3, which ought to be the sums of the coefficients of the several partial products; and they are found to be so.

39. When the factors contain different powers of the same quantity, it is advantageous, for the purpose of making like quantities in the several partial products fall as much as possible in the same columns, to arrange the terms according to the indices of those powers; that is, to place them so that each index may be less, or that each may be greater, than the succeeding one. Thus, if the given factors were $4a^2x+2x^3-ax^2$ and $2ax-2a^2+3x^2$, either of the arrangements in the margin might be used; and the results would be the same, only reversed in order. In the first way, the terms are arranged according to the *descending* powers of x, and in the second according to the *ascending* ones. With regard to a, the arrangement is the reverse of this.

$$\begin{array}{l} 2x^3-ax^2+4a^2x \\ 3x^2+2ax-2a^2 \\ \\ 4a^2x-ax^2+2x^3 \\ -2a^2+2ax+3x^2 \end{array}$$

Exercises.—Find the products of the following quantities:—

1. $5ax$ and $3ay$. *Ans.* $15a^2xy$.
2. v^{n+1} and v^{n-1}. *Ans.* v^{2n}.
3. v^{m+p} and v^{n-p}. *Ans.* v^{m+n}.
4. x^5 and x^5. *Ans.* x^{10}.
5. x^n and x^n. *Ans.* x^{2n}.
6. x^2y^3 and x^5y^2. *Ans.* x^7y^5.
7. $x^3-3x^2y+3xy^2$ and axy. *Ans.* $ax^4y-3ax^3y^2+3ax^2y^3$.
8. $1-2x+3x^2-4x^3$ and $1+x$. *Ans.* $1-x+x^2-x^3-4x^4$.
9. $x^2+2ax+a^2$ and $x^2-2ax+a^2$. *Ans.* $x^4-2a^2x^2+a^4$.
10. $v-2x$ and $2x-3z$. *Ans.* $2vx-4x^2-3vz+6xz$.
11. Required the third power of $2x+3y$.
Ans. $8x^3+36x^2y+54xy^2+27y^3$.[*]
12. Find the fifth power of $2x-3y$.
Ans. $32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5$.
13. Required the continual product of $a+b+c$, $-a+b+c$, $a-b+c$, and $a+b-c$. *Ans.* $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$.
14. Find the product of $x^2-y^2+z^2-v^2$ and $x^2+y^2-z^2-v^2$.
Ans. $x^4-y^4-z^4+v^4-2v^2x^2+2y^2z^2$.
15. Find the continual product of $2x-y$, $2x+y$, and $4x^2+y^2$.
Ans. $16x^4-y^4$.

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[*] This is obtained by multiplying $2x+3y$ by itself, to find its second power; and that result by $2x+3y$. The next exercise may be wrought in different ways. Thus, the successive powers, the second, third, fourth, and fifth, may be found by repeated multiplications by $2x-3y$; or, when the second power is found, it may be multiplied by itself to find the fourth, and that by $2x-3y$; or, lastly, the third power may be found, and may be multiplied by the second. Let the student compare the third power of $2x-3y$ with the answer to Exer. 11.

16. Square $1+x+x^2+x^3+x^4$.

Ans. $1+2x+3x^2+4x^3+5x^4+4x^5+3x^6+2x^7+x^8$.

40. *When the factors consist of powers of the same quantity,* it is sufficient to perform the operations by means of the coefficients alone, and in the results so obtained to introduce the proper powers of that quantity.

Exam. 7. Required the product of $2x^4-5x^3-2x^2+3x-4$ and $3x^2-2x+1$.

Here, we write out the coefficients of the multiplicand, 2, -5, -2, 3, and -4, in their proper order, and with their proper signs; and, below the leading terms, 2, -5, and -2, we place 3, -2, and 1, the coefficients of the multiplier. We then multiply successively by 3, -2, and 1, setting the first term of each product under the coefficient of the multiplier which produces it, and attending to the signs, according to § 36. In the last place, we find the sums of the several columns by the rules for addition; and, as the product of x^4 and x^2, the highest powers of x, is x^6, we insert x^6 after the first coefficient 6, and the successive inferior powers after the others till we come to -4, to which no power of x is annexed, as it is the product of the terms not containing x. The answer, therefore, is $6x^6-19x^5+6x^4+8x^3-20x^2+11x-4$.

$$\begin{array}{rrrrrrr}
2 & -5 & -2 & 3 & -4 & & \\
3 & -2 & 1 & & & & \\
\hline
6 & -15 & -6 & 9 & -12 & & \\
 & -4 & 10 & 4 & -6 & 8 & \\
 & & 2 & -5 & -2 & 3 & -4 \\
\hline
6 & -19 & 6 & 8 & -20 & 11 & -4.
\end{array}$$

Exam. 8. Find the product of $4x^6-2x^5-5x^3+3x^2$ and $2x^4-x^2-4x$.

Here, the work proceeds as in the last example, except that, as x^4 does not occur in the multiplicand, we write 0 for its coefficient. In like manner we put 0 as the coefficient of x^3 in the multiplier. The answer is found to be $8x^{10}-4x^9-4x^8-24x^7+14x^6+5x^5+17x^4-12x^3$.

$$\begin{array}{rrrrrrrr}
4 & -2 & 0 & -5 & 3 & & & \\
2 & 0 & -1 & -4 & & & & \\
\hline
8 & -4 & 0 & -10 & 6 & & & \\
 & & -4 & 2 & 0 & 5 & -3 & \\
 & & & -16 & 8 & 0 & 20 & -12 \\
\hline
8 & -4 & -4 & -24 & 14 & 5 & 17 & -12.
\end{array}$$

41. The same principle may be employed with equal advantage when the terms of the given factors are products of the powers of more quantities than one, provided the indices of each set of these powers successively increase or diminish by equal differences.

Exam. 9. Multiply $x^4-3ax^3+4a^2x^2$ by $x^3+3ax^2-a^2x$.

Here, the indices of x decrease in both factors from term to term by unity, and those of a increase by the same. In the final product, the coefficient of the second term, ax^6, is 0, and therefore that term is wanting.

$$\begin{array}{rrrrr}
1 & -3 & 4 & & \\
1 & 3 & -1 & & \\
\hline
1 & -3 & 4 & & \\
 & 3 & -9 & 12 & \\
 & & -1 & 3 & -4 \\
\hline
1 & 0 & -6 & 15 & -4,\text{ or}
\end{array}$$

$$x^7-6a^2x^5+15a^3x^4-4a^4x^3.$$

Exam. 10. Required the product of $2xy^5+x^2y^3-3x^3y$, and $x^2y^6-3x^3y^4+x^4y^2$.

In this example, the indices of x increase by 1 as common difference, while those of y diminish by 2 from term to term. The first term of the product is plainly $2x^3y^{11}$: and the slightest consideration of the mode in which the terms would be obtained in the uncontracted process, will show, that, in this and all similar questions, the indices in the product will follow the same law as in the factors. Accordingly, the answer is $2x^3y^{11}-5x^4y^9-4x^5y^7+10x^6y^5-3x^7y^3$.

$$\begin{array}{rrrrr}
2 & 1 & -3 & & \\
1 & -3 & 1 & & \\
\hline
2 & 1 & -3 & & \\
 & -6 & -3 & 9 & \\
 & & 2 & 1 & -3 \\
\hline
2 & -5 & -4 & 10 & -3
\end{array}$$

Exam. 11. Multiply $xy+2x^3y^2+x^5y^3$ by $x^2y-4x^4y^2+x^6y^3$.

Here, in the factors, the indices of x increase by a common difference of 2, and those of y by 1. Therefore, after the coefficients of the product are found, and after it is seen that the first term is x^3y^2, we determine the powers of x and y in the remaining terms by the same law; and the product is $x^3y^2-2x^5y^3-6x^7y^4-2x^9y^5+x^{11}y^6$.

$$\begin{array}{rrrrr}
1 & 2 & 1 & & \\
1 & -4 & 1 & & \\
\hline
1 & 2 & 1 & & \\
 & -4 & -8 & -4 & \\
 & & 1 & 2 & 1 \\
\hline
1 & -2 & -6 & -2 & 1
\end{array}$$

For gaining practice in this extremely simple and easy method of performing multiplication, the learner may work according to it the 5th and 6th examples, and the 8th, 9th, 11th, 12th, 15th, and 16th of the last set of exercises; and also most of the following exercises.

Exercises.—Find the products of the following quantities:—

17. $x-2x^2+3x^3$ and $4x^4+5x^5-6x^6$.

Ans. $4x^5-3x^6-4x^7+27x^8-18x^9$.

18. $5y^5-7y^4-8y^3+3y^2+y$ and $7y-8$.

Ans. $35y^6-89y^5+85y^3-17y^2-8y$.

19. a^3-2a^2+3 and a^3+2a-3.
Ans. $a^6-2a^5+2a^4-4a^3+6a^2+6a-9$.

20. $v^4-4av^3+6a^2v^2-4a^3v+a^4$ and $v^3-3av^2+3a^2v-a^3$.
Ans. $v^7-7av^6+21a^2v^5-35a^3v^4+35a^4v^3-21a^5v^2+7a^6v-a^7$.

21. $x^3-a^2x+2a^3$ and $x^2-ax+2a^2$.
Ans. $x^5-ax^4+a^2x^3+3a^3x^2-4a^4x+4a^5$.

22. Required the second and third powers of $x^2-2ax-3a^2$.
Ans. $x^4-4ax^3-2a^2x^2+12a^3x+9a^4$,
and $x^6-6ax^5+3a^2x^4+28a^3x^3-9a^4x^2-54a^5x-27a^6$.

23. Find the continual product of $x-1$, $x+2$, $x+4$, and $x-5$.
Ans. $x^4-23x^2-18x+40$.

24. Multiply $1-x+x^2-x^3+x^4-x^5$ by $1+x+x^2$.
Ans. $1+x^2-x^3+x^4-x^5-x^7$.

25. Multiply $1+x+x^2+x^3+x^4+x^5$ by $1-x+x^2$.
Ans. $1+x^2+x^3+x^4+x^5+x^7$.

26. Mult. $1+x+x^2+x^3+x^4+x^5$ by $1-x+x^2-x^3+x^4-x^5$.
Ans. $1+x^2+x^4-x^6-x^8-x^{10}$.

27. Multiply $a+2b+c$ by $a-c$. *Ans.* $a^2+2ab-2bc-c^2$.

28. Find the continual product of $xy-1$, $xz-1$, and $yz-1$.
Ans. $x^2y^2z^2-x^2yz-xy^2z-xyz^2+xy+xz+yz-1$.

29. Find the continual product of x^2+yz, y^2+xz, and z^2+xy.
Ans. $2x^2y^2z^2+x^3y^3+x^3z^3+y^3z^3+xyz^4+xzy^4+yzx^4$.

30. Multiply $a^2+b^2+c^2-ab-ac-bc$ by $a+b+c$.
Ans. $a^3+b^3+c^3-3abc$.

DIVISION.

42. ***When quantities are divided by a positive term, their signs are retained in the quotient; but when by a negative one, they are changed:*** or, as the rule is generally expressed, ***in division, like signs produce*** plus ***in the quotient, and unlike signs*** minus.

Since, by the nature of division, the quotient is to be such, that if it be multiplied by the divisor the product will be the dividend, the rule now given is easily proved on this principle. Thus, in the first line in the margin, we have every variety of signs in dividing by the positive quantity a, one of the dividends being positive and the other negative; and the second part exhibits the same variety in dividing by the negative quantity $-a$. Then, in dividing ab and $-ab$, by a, the quotients are respectively b and $-b$, and they can be nothing else; as, by the nature

$$\begin{array}{ll} a)\underline{ab} & a)\underline{-ab} \\ \quad b & \quad -b \\ -a)\underline{ab} & -a)\underline{-ab} \\ \quad -b & \quad b \end{array}$$

of multiplication, the products of these, and only these, by a, will be the respective dividends; and these quotients, which are produced by dividing by the positive quantity a, have each the sign of the dividend from which it is obtained. In the second part, the quotients, found on the same principle, are $-b$ and b; which are produced by dividing by the negative quantity $-a$, and which have respectively the signs that are opposite to those of the dividends which produce them.

43. *When the divisor is a simple quantity, and is a factor of the dividend, the quotient is obtained from the dividend by removing that factor.* Thus, if xy be divided by y, the quotient is x; and, if xyz be divided by xy, the quotient is z; as in each instance the product of the quotient by the divisor is the dividend.

44. *One term may be divided by another*, by placing the dividend over the divisor so as to form a fraction; and if there be a factor common to the numerator and denominator, it may be removed from both to simplify the quotient.*

Thus, if a be divided by b, the quotient (§ 8.) is $\frac{a}{b}$; and, if abc be divided by abd, the quotient is $\frac{abc}{abd}$, or, simply, $\frac{c}{d}$, by suppressing the common factor ab.

To show that $\frac{abc}{abd}=\frac{c}{d}$, let $\frac{abc}{abd}=q$, and multiply by abd; then, $abc=abdq$. Divide these equals by ab; then (§ 43.) $c=dq$. Hence, by dividing by d, and putting the value of q, so found, equal to its former value, we have the proof.

45. *To divide one power of a quantity by another;* take the index of the divisor from that of the dividend, and place the remainder as index to the same quantity.

Thus, if a^7 be divided by a^3, the quotient is a^4. For $a^7=a^3.a^4$; and dividing this by a^3, we get (§ 43.) a^4. The rule might also be illustrated by means of § 38.

So, likewise, $x^6 \div x=x^5$; $x^m \div x=x^{m-1}$; $x^m \div x^n=x^{m-n}$, &c.

---

* After a little practice, the student will be able, in general, to dispense with the direct use of this rule; as, in most instances, he will readily discover the quotient by inspection. It is plain, from the note to § 36., that in dividing a compound quantity by a simple one the terms of the former are to be separately divided by the latter. Thus, if $ax-ay$ be divided by a, the quotient is $x-y$.

Examples in which the Divisor is a Simple Quantity.

Exam. 1.

$$3a)\underline{6ax-9ay-3az+12av}$$
$$2x-3y-z+4v$$

Exam. 2.

$$ab^2)\underline{4a^4b^3-a^6b}$$
$$4a^3b-\frac{a^5}{b}$$

Exam. 3.

$$2x^3)\underline{4x^6-a^3x^3+2a^4x^2}$$
$$2x^3-\frac{a^3}{2}+\frac{a^4}{x}$$

In the second of these examples we may divide a^6b by ab^2 either according to § 44.; or we may divide by a, which gives a^5b; then, for dividing by b^2, we may divide twice in succession by b, which can all be done mentally without difficulty. In Exam. 3., the second term of the answer might also be written $\frac{1}{2}a^3$. In like manner, instead of $\frac{3x}{4}$, we might write $\frac{3}{4}x$; and so in all similar cases. See *Arithmetic*, p. 78.

Exercises.

1. Divide ab^2c^3 by abc, and x^6y^5 by x^2y. *Ans.* bc^2 and x^4y^4.
2. Divide x^{m+n} and x^{m-n} each by x^n. *Ans.* x^m and x^{m-2n}.
3. Divide $a^3x^4-a^4x^3+a^5x^2$ by a^3x^2. *Ans.* x^2-ax+a^2.
4. Divide $2a^5-3a^4y-6a^3y^2$ by $3a^2$. *Ans.* $\frac{2}{3}a^3-a^2y-2ay^2$.

46. When in the application of the rule in § 45., the index of the divisor is greater than that of the dividend, the resulting index is negative. Thus, $x^5\div x^8=x^{-3}$, and $x^m\div x^{m+n}=x^{-n}$. We thus obtain expressions of a new kind; which, however, by an extension of the meaning of the term *power*, are still called *powers*, and, for the sake of distinction, *powers with negative indices*. To interpret these in reference to quantities already explained, we have only to perform the division according to § 44. In this way, for x^{-3} we get $\frac{x^5}{x^8}$, or $\frac{1.x^5}{x^3.x^5}$, or finally, $\frac{1}{x^3}$; while, for x^{-n}, we have $\frac{x^m}{x^{m+n}}$, or $\frac{1.x^m}{x^m.x^n}$, or $\frac{1}{x^n}$. It thus appears, that *a power having a negative index is the reciprocal* of the same quantity raised to the*

---

* *The reciprocal of a quantity* (Arithmetic, p. 94.) *is the quotient obtained by dividing unity by the quantity;* and hence it follows, that two numbers or quantities whose product is 1, are reciprocals of each other. Thus, $\frac{1}{2}$ is the reciprocal of 2, and 2 the reciprocal of $\frac{1}{2}$.

power which has for its index the same index taken positively. Thus, $3^{-1}=\frac{1}{3}$; also, $2^{-3}=\frac{1}{8}$, since, $2^3=8$.

47. Since $x^{-n}=\frac{1}{x^n}$, multiply both by a; then $ax^{-n}=\frac{a}{x^n}$.* Hence, *a quantity with a negative index may be made denominator, with a positive index, to the factor connected with it, taken as numerator; and a denominator may be written along with the numerator, if the sign of its index be changed.*

Thus, $y^{-1}=\frac{1}{y}$, $cx^{-3}=\frac{c}{x^3}$, $a(x^2+b^2)^{-2}=\frac{a}{(x^2+b^2)^2}$, &c.; and $\frac{a}{x}=ax^{-1}$, $\frac{ab^2}{x^2y^3}=ab^2x^{-2}y^{-3}$, $\frac{1}{x-a}=(x-a)^{-1}$, $\frac{b}{a^2-x^2}=b(a^2-x^2)^{-1}$, &c.

48. When a quantity is divided by itself the quotient is unity. Now, if x^n be divided by x^n according to § 45., the quotient is x^0. Hence, *a quantity having zero as index is equivalent to unity;*

---

* By multiplying this by x^n, we get $ax^nx^{-n}=a$; and hence we obtain $ax^n=\frac{a}{x^{-n}}$. Combining this with what is established in the text, since ax^n and ax^{-n} may be regarded as numerators to the denominator 1, we see, that, *in fractions, any quantity may be removed from the numerator to the denominator, or from the denominator to the numerator, if the sign of its index be changed;* a principle analogous to the transposition of quantities in the resolution of equations. Except for generalising the principle, however, it is unnecessary to consider fractions having quantities with negative indices in their denominators, as such fractions do not occur in practice.

By means of this mode of notation, fractions, especially when their terms are simple quantities, may be written in a condensed form, and often more neatly than in the common mode. Thus, instead of $\frac{p}{q}$, $\frac{p^2}{q^2}$, &c., we may write pq^{-1}, p^2q^{-2}, &c. Such expressions, therefore, will be occasionally used in what follows; and the learner has merely to recollect that every such expression may be put under the usual form by taking the quantity with the positive index as the numerator of a fraction, and the other with the sign of its index changed, as its denominator.

The extension of the meaning of the term *power* established above, as well as another extension still to be given, is of much importance in preventing the necessity for multiplying rules, as the rules for the management of powers of the primitive kind are applicable with regard to the others. Thus, to multiply x^{-m} by x^{-n}, we simply add the indices, as in § 38., thus getting for product x^{-m-n}. This will follow from what will be established in a subsequent section regarding the multiplication of fractions; as, according to what will there be shown, the product of x^{-m} and x^{-n} will be a fraction, having 1 for numerator, and x^{m+n} for denominator, which fraction is equivalent to x^{-m-n}.

and therefore, conversely, unity may be replaced by such a quantity, and by this means the trace of any quantity may be retained or introduced. Thus, instead of x^4+ax^2+b, we may write $x^4+ax^2+bx^0$; and, in connexion with § 47., $x-2+\frac{3}{x}$ may be written $x-2x^0+3x^{-1}$.

49. In dividing by a *compound quantity*, the process, in the common mode, agrees so entirely in form and principle with the operations in " long division ", in common arithmetic, that, after what has been already established, we may proceed at once with examples, no rule being necessary, except that, for facilitating the operation, the terms ought to be arranged in the manner pointed out in § 39.

Exam. 4. Divide $12x^5-17x^4-2x^3+18x^2-9x$ by $4x^2-3x$.

$$\begin{array}{l|l}
12x^5-17x^4-2x^3+18x^2-9x & 4x^2-3x\,\text{*} \\ \cline{2-2}
\underline{12x^5-9x^4} & 3x^3-2x^2-2x+3 \\
-8x^4-2x^3 & \\
\underline{-8x^4+6x^3} & \\
-8x^3+18x^2 & \\
\underline{-8x^3+6x^2} & \\
12x^2-9x & \\
\underline{12x^2-9x} & \\
0 &
\end{array}$$

In this operation, we find (§§ 42. and 43.) that if $12x^5$ be divided by $4x^2$, the quotient is $3x^3$; which, therefore, is the first term of the quotient. By multiplying the divisor by this, we get $12x^5-9x^4$; and, taking this from the two leading terms of the dividend, we get for remainder $-8x^4$, to which we annex $-2x^3$, the next term of the dividend. Then, dividing (§§ 42. and 43.) $-8x^4$ by $4x^2$, we get $-2x^2$, the next term of the quotient. Mul-

* In the form of operation here adopted, the divisor is placed to the right of the dividend, and the quotient below it, as is done by many of the Continental writers. This arrangement has the advantage, both in algebra and in common arithmetic, of keeping the quantities, of which, in the course of the operation, the products are to be found, much nearer each other, than they are kept in the arrangement adopted by writers in this country; and, in this way, the breadth occupied by the quantities in the operation is conveniently diminished. Should any person, however, from habit, or any other reason, prefer the method usually employed in this country, it is only necessary to place the divisor, dividend, and quotient as follows, and so in other similar examples:

$$4x^2-3x)12x^5-17x^4-2x^3+18x^2-9x(3x^3-2x^2-2x+3.$$

tiplying the divisor by this, we obtain $-8x^4+6x^3$; and, taking this product from $-8x^4-2x^3$, we find $-8x^3$, to which we annex $18x^2$, the next term of the dividend. Proceeding in a similar manner, we get $-2x$ for the next term of the quotient, and $12x^2$ as remainder, to which we annex $-9x$, the remaining term of the dividend. This, by a like process, gives 3 for the remaining term of the quotient, with no remainder.*

Exam. 5.

$$\begin{array}{l|l}
8ax^6-2a^2x^5+4a^3x^4+7a^4x^3+a^5x^2-6a^6x & 2x^3-2ax^2+3a^2x \\ \cline{2-2}
\underline{8ax^6-8a^2x^5+12a^3x^4\qquad\qquad\qquad} & 4ax^3+3a^2x^2-a^3x-2a^4 \\
\multicolumn{2}{l}{\qquad\quad 6a^2x^5-8a^3x^4+7a^4x^3} \\
\multicolumn{2}{l}{\qquad\quad \underline{6a^2x^5-6a^3x^4+9a^4x^3}} \\
\multicolumn{2}{l}{\qquad\qquad\qquad -2a^3x^4-2a^4x^3+a^5x^2} \\
\multicolumn{2}{l}{\qquad\qquad\qquad \underline{-2a^3x^4+2a^4x^3-3a^5x^2}} \\
\multicolumn{2}{l}{\qquad\qquad\qquad\qquad\qquad -4a^4x^3+4a^5x^2-6a^6x} \\
\multicolumn{2}{l}{\qquad\qquad\qquad\qquad\qquad \underline{-4a^4x^3+4a^5x^2-6a^6x}} \\
\multicolumn{2}{l}{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \qquad\quad 0}
\end{array}$$

In this example the divisor is $2x^3-2ax^2+3a^2x$, and the dividend $8ax^6-2a^2x^5+4a^3x^4+7a^4x^3+a^5x^2-6a^6x$; and the quotient is found to be $4ax^3+3a^2x^2-a^3x-2a^4$.

Exam. 6. Divide x^3 by $x+2a$.

Here, after finding three terms of the quotient in the usual manner, we have the remainder $-8a^3$. As this does not contain x, the leading term of the divisor, the work is to be regarded as terminated, unless, as will be explained hereafter, we choose to work for a series of fractional quantities. After

$$\begin{array}{r|l}
x^3\qquad\quad & x+2a \\ \cline{2-2}
\underline{x^3+2ax^2} & x^2-2ax+4a^2-\dfrac{8a^3}{x+2a} \\
-2ax^2 & \\
\multicolumn{2}{l}{\quad\underline{-2ax^2-4a^2x}} \\
\multicolumn{2}{l}{\qquad\qquad\quad 4a^2x} \\
\multicolumn{2}{l}{\qquad\qquad\quad \underline{4a^2x+8a^3}} \\
\multicolumn{2}{l}{\qquad\qquad\qquad\quad -8a^3}
\end{array}$$

---

* The correctness of the work would be proved by multiplying the quotient by the divisor, as the product would be the same as the dividend. The learner ought to accustom himself to prove such operations in this way, both for the purpose of gaining expertness in multiplication, and for assuring himself of the correctness of the answer. A convenient verification, so far as regards the coefficients, will be obtained on the plan pointed out in the note in p. 24.; as it is plain, that if the sum of the coefficients of the divisor be multiplied by the sum of those of the quotient, and the sum of the coefficients of the remainder be added to the product so found, the result should be equal to the sum of the coefficients of the dividend. This method will fail, when the sums of the coefficients of the divisor, dividend, and remainder are each equal to nothing.

the three terms, therefore, we write, as in common arithmetic, $8a^3$ with the divisor, $x+2a$, as denominator, prefixing the sign of the remainder. The proof will stand as in the margin, the remainder, $-8a^3$ being placed in the first line, so as to be added with the two partial products.

$$\begin{array}{l}
x^2-2ax\;+4a^2\\
\underline{x\;+2a\qquad\qquad\qquad\quad}\\
\hfill -8a^3\\
x^3-2ax^2+4a^2x\\
\underline{\qquad\;\; 2ax^2-4a^2x+8a^3}\\
x^3
\end{array}$$

Exam. 7. Divide $x^3-a^2x+2a^3$ by x^2-ax+a^2.

$$\begin{array}{r|l}
x^3-a^2x+2a^3 & x^2-ax+a^2\\ \cline{2-2}
\underline{x^3-ax^2+a^2x} & x+a-\dfrac{a^2x-a^3}{x^2-ax+a^2}.\\
ax^2-2a^2x+2a^3 & \\
\underline{ax^2-\;a^2x+\;a^3} & \\
-\;a^2x+\;a^3 &
\end{array}$$

In this example, after finding the two terms, x and a, of the quotient, we have remaining $-a^2x+a^3$; in which the highest index of x is less than its highest index in the divisor, so that the operation cannot be continued farther without giving origin to fractions. In such cases it is usual, as in the last example, to consider the work as terminated; and, as in common arithmetic, to place the remainder over the divisor, and to annex the fraction so found to the part of the quotient already obtained, to complete it. Hence, in the present instance, the quotient might be written, $x+a+\frac{-a^2x+a^3}{x^2-ax+a^2}$. When, however, as here, the leading term of the remainder is negative, it is usual, as has been done here, *though perhaps not preferable*, to change the signs of all the terms of the remainder, and to prefix the sign — to the fractions.*

Exercises.

	Dividends.	Divisors.	Quotients.
5.	$3a^5b^3-10a^4b^4+8a^3b^5$	$3a^3b^2-4a^2b^3$	a^2b-2ab^2
6.	$x^5-13a^2x^3+12a^3x^2$	$x^2+3ax-4a^2$	x^3-3ax^2
7.	$x^4-9x^2y^2+12xy^3-4y^4$	$x^2-3xy+2y^2$	$x^2+3xy-2y^2$
8.	$x^4-6x^3+5x^2+12x+4$	x^2-3x-2	x^2-3x-2
9.	x^2+a^2	$x-a$	$x+a+\frac{2a^2}{x-a}$
10.	$a^2-b^2+2bc-c^2$	$a-b+c$	$a+b-c$

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* That the fractional parts in the two modes of expression are equivalent, will appear from considering, that, as is plain from the note to § 36., $\frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}$, and that, therefore,

$$d-\frac{a-b}{c}=d-\frac{a}{c}+\frac{b}{c}=d+\frac{-a}{c}+\frac{b}{c}=d+\frac{-a+b}{c}.$$

11. Divide $81x^5+24x^2$ by $3x^2+2x$. *Ans.* $27x^3-18x^2+12x$.

12. Divide x^3+a^3 by x^2+a^2. *Ans.* $x-\dfrac{a^2x-a^3}{x^2+a^2}$.

13. Divide $10y^6-23ay^5+4a^2y^4$ by $5y^4-4ay^3+a^2y^2$.

Ans. $2y^2-3ay-2a^2-\dfrac{5a^3y^3-2a^4y^2}{5y^4-4ay^3+a^2y^2}$.

50. The method of ***detached coefficients***, explained in §§ 40. and 41. in reference to multiplication, may often be employed with great advantage in division. This method will be understood from the following examples.

Exam. 8. Divide $3x^4+3x^3-4x^2+3x-5$ by $x+2$.

$$\begin{array}{rrrrr|r} 3 & 3 & -4 & 3 & -5 & -2 \\ & -6 & 6 & -4 & 2 & \\ \hline & -3 & 2 & -1 & -3, \text{ or} & \end{array}$$

$$3x^3-3x^2+2x-1-\frac{3}{x+2}.$$

$$\begin{array}{l|l} 3x^4+3x^3-4x^2+3x-5 & x+2 \\ \underline{(3x^4)+6'x^3} & 3x^3-3x^2+2x-1. \\ \qquad -3x^3-4x^2 & \\ \quad \underline{(-3x^3)-6'x^2} & \\ \qquad\qquad 2x^2+3x & \\ \qquad\quad \underline{(2x^2)+4'x} & \\ \qquad\qquad\quad -1x-5 & \\ \qquad\qquad \underline{(-1x)-2'} & \\ \qquad\qquad\qquad -3 & \end{array}$$

In this example the coefficient of the first term of the divisor is unity; and this is generally so in the cases in which this method is most advantageously employed in practice, such as in the resolution of the higher equations. In this case, we omit the 1, and write the next number, 2, with its sign changed, in the place of the divisor. We write out also the coefficients of the dividend in succession. We then multiply -2 by the first of these coefficients, and, placing the product, -6, under the second coefficient, we add it to that coefficient. The sum, -3, is then multiplied into -2, and the product, 6, is set under the next coefficient, and added to it. The result 2 is in like manner multiplied into -2, the product set under the next coefficient, and added to it; and thus we proceed as long as there are coefficients.

The reason of the process will be understood from comparing it with the annexed work at full length, with which, in fact, it is virtually identical, the form and arrangement being merely changed, and every thing omitted that can be dispensed with. Thus, it will be seen, that the four terms enclosed thus $(3x^4)$, &c., are omitted; and that the second line in the first process consists of the numbers marked with accents in the other, with their signs changed; while the numbers standing below the accentuated ones, are the same as those in the third line of the short process. The changing of 2 into -2 converts the subtractions in the common method into additions; and it will be seen from the full process, that the first of the given coefficients, and the numbers in the last line of the short process, except the last, which is the remainder, are the coefficients of the quotient.*

Ex. 9. Divide $2x^6-10x^5+19x^4-8x^3+x^2$ by $x^4-3x^3+2x^2$.

$$\begin{array}{rrrrr|l} 2 & -10 & 19 & -8 & 1 & 3-2 \\ & 6 & -4 & 8 & -6 & \\ \hline & -4 & -12 & 9 & -5 & \\ & & 3 & 9 & & \end{array}$$

$$2x^2-4x+3+\frac{9x^3-5x^2}{x^4-3x^3+2x^2},\ \textit{Ans.}$$

$$\begin{array}{rrrrr|l} 2x^6 & -10x^5 & +19x^4 & -8x^3 & +x^2 & x^4-3x^3+2x^2 \\ 2x^6 & -\ 6x^5 & +\ 4x^4 & & & \overline{2x^2-4x+3} \\ \hline & -\ 4x^5 & +15x^4 & -8x^3 & & \\ & -\ 4x^5 & +12x^4 & -8x^3 & & \\ \hline & & 3x^4 & 0 & +\ x^2 & \\ & & 3x^4 & -9x^3 & +6x^2 & \\ \hline & & & 9x^3 & -5x^2 & \end{array}$$

In working this example by the contracted process, we write the coefficients of the dividend in succession, and after them the second and third coefficients of the divisor with their signs changed. The two latter are there multiplied by 2, the first coefficient of the dividend, and the products, 6 and -4, are set under -10 and 19, the

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* The learner will perhaps be assisted in perceiving the identity of the two processes by the following considerations. From the suppression of the powers of x, the second coefficient 3 in the first process, means $3x^3$, the second term in the other, and the number -6 below it, means $-6x^3$; and the adding of $-6x^3$ in the one process is the same as the subtracting of $6x^3$ (the same quantity with its sign changed) in the other. The next coefficient -4 means $-4x^2$, the third term in the second process, which term, according to the usual mode, is written over again after $-3x^3$. The number 6 also, below -4, means $6x^2$; and the adding of $6x^2$ to $-4x^2$ in the first process is the same as, in the other process, the subtracting of $-6x^2$ (the same quantity with its sign changed) from $-4x^2$ in its new position; and thus the rest of the process might be illustrated. Should any difficulty still be felt, it may perhaps be removed by introducing, for the sake of illustration, the powers of x, as in the margin. From considering the full process, it will be seen, that, to get the terms of the quotient, each of the quantities, $3x^4$, $-3x^3$, $2x^2$, and $-1x$, must be divided by x, the first term of the divisor.

$$\begin{array}{rrrrr|l} 3x^4 & +3x^3 & -4x^2 & +3x & -5 & -2 \\ & -6x^3 & 6x^2 & -4x & 2 & \\ \hline & -3x^3 & 2x^2 & -1x & -3 & \end{array}$$

next coefficients. In the next place, 6 and -10 are added together, and the sum, -4, being multiplied into 3 and -2, the products, -12 and 8, are placed in the two columns following that in which the multiplier, -4, stands. The quantities in the column headed by 19, are then added, and the sum, 3, is multiplied into 3 and -2. The products, 9 and -6, are placed in the columns next after that in which the multiplier 3 stands, and the quantities in these two columns are added together, which terminates the work. To express the answer in the usual form, we see that the first term must contain x^2, since the index in the first term of the dividend is greater by 2 than in that of the divisor. Hence, to the first coefficient 2, we annex x^2, and to -4, the number found in the second column, we affix x. The 3 in the third column is a mere number (or the coefficient of x^0). The remaining figures, 9 and -5, are the coefficients of the remainder; the first that of x^3, the power next lower than the highest in the divisor, and the next that of x^2.

The explanation of the process will be obtained, as in the last example, by comparing it with the work at full length. It will thus appear, that, in the arithmetical sense, 6 is taken from -10 in the second column of both processes, leaving the same remainder, -4. In the third column, 4 and 12 are taken from 19 in the first process, while, in the second mode, 4 is taken from 19, and 12 from the remainder, which is manifestly equivalent: and in this way the process may always be illustrated.

Exam. 10. Divide $2x^6-8x^4+20x^3$ by x^3+2x^2-x+3.

$$\begin{array}{rrrrrrrl}
2 & 0 & -8 & 20 & 0 & 0 & 0 & \mid -2+1-3\\
 & \underline{-4} & 2 & -6 & 12 & -6 & \underline{-18} & \\
 & -4 & 8 & -4 & 2 & \underline{6} & -18 & \\
 & & \overline{2} & -4 & -12 & 0 & & \\
 & & & \overline{6} & \overline{2} & & &
\end{array}$$

$$2x^3-4x^2+2x+6+\frac{2x^2-18}{x^3+2x^2-x+3}. \quad \textit{Ans.}$$

In this example ciphers are employed as if the dividend were written under the form, $2x^6+0x^5-8x^4+20x^3+0x^2+0x+0$; the principle being, that the dividend should be carried as low with regard to powers, as the divisor.

The work of the next example is given at full length. It might be slightly contracted, however, by omitting the ciphers in the body of the operation.

Exam. 11. Divide x^5+2x^4 by x^2+2.

$$\begin{array}{rrrrrr|l}
1 & 2 & 0 & 0 & 0 & 0 & 0-2 \\
 & \underline{0} & -2 & -4 & 4 & \underline{8} & \\
 & 2 & \underline{0} & \underline{0} & \underline{0} & 8 & \\
 & & -2 & -4 & 4 & &
\end{array}$$

$$x^3+2x^2-2x-4+\frac{4x+8}{x^2+2}. \quad \textit{Ans.}$$

Exam. 12. Divide $x^4+3ax^3+8a^4$ by $x+2a$.

$$\begin{array}{rrrrr|l}
1 & 3 & 0 & 0 & 8 & -2 \\
 & \underline{-2} & \underline{-2} & \underline{4} & \underline{-8} & \\
 & 1 & -2 & 4 & 0 &
\end{array}$$

$$x^3+ax^2-2a^2x+4a^3. \quad \textit{Ans.}^*$$

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* The annexed example will illustrate the manner in which the method of detached coefficients may be employed, when the coefficient of the first term of the divisor is not unity. In this, the first coefficient in the divisor being 2, we multiply the successive coefficients of the dividend by 1, 2, 4, 8, 16, and 32 (2^0, 2^1, 2^2, 2^3, &c., the successive powers of the coefficient 2); and, omitting the first term of the divisor, we multiply by the same numbers, changing the signs of the products. The work then proceeds in the usual manner, and we get 1, −7, 17, −39, 114, and 28. The first three of these divided by 2, 4, and 8 (the powers of 2) are the coefficients of x^2, &c.; and the remaining three, −39, &c., divided by 1, 2, and 4, are the coefficients of x^2, &c., in the numerator of the fraction arising from the remainder. Then, to get the denominator of this fraction, we write the given divisor below the numerator, and divide by 8, the same power by which 17 is divided. In the same way we may proceed in every similar case, using the first coefficient of the divisor, and the powers of that coefficient in the same manner in which 2 and its powers have been used here.

Example.

$$\begin{array}{rrrrrr|rrr}
x^5-3x^4 & & +\ x^3 & +\ 3x^2 & -\ x & +\ 3 & 2x^3 & +\ x^2-3x & +1 \\
1 & 2 & 4 & 8 & 16 & 32 & 1 & 2 & 4 \\
\hline
1 & -6 & 4 & 24 & -16 & 96 & -1 & 6 & -4 \\
 & \underline{-1} & 6 & -4 & 28 & \underline{-68} & & & \\
 & -7 & \underline{7} & -42 & \underline{102} & 28|4 & & & \\
 & & 17 & \underline{-17} & 114|2 & & & & \\
 & & & -39 & & & & &
\end{array}$$

$$\frac{x^2}{2}-\frac{7x}{4}+\frac{17}{8}-\frac{1}{8}\cdot\frac{39x^2-57x-7}{2x^3+x^2-3x+1}. \quad \textit{Ans.}$$

To illustrate this process, let $2x$ be denoted by y. Then $x=\frac{y}{2}$; by the

Exercises.

[Of the exercises already given, Nos. 6, 7, 8, 9, and 12. may all be wrought by the method of detached coefficients, as already explained; and the greater part of the following may be done most readily in the same way. When more than two quantities are concerned, as in Exer. 20. and others, it is generally better to employ the complete terms.]

	Dividends.	Divisors.	Quotients.
14.	$x^5-3x^4+x^3+x^2-3x+1$	$x+1$	$x^4-4x^3+5x^2-4x+1$
15.	$x^5-x^4-2x^3-2x^2+5x-2$	$x-2$	x^4+x^3-2x+1

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substitution of which in the dividend and divisor, they become

$$\frac{y^5}{2^5}-\frac{3y^4}{2^4}+\frac{y^3}{2^3}+\frac{3y^2}{2^2}-\frac{y}{2}+3 \,\Big|\, \frac{2y^3}{2^3}+\frac{y^2}{2^2}-\frac{3y}{2}+1;$$

whence, by multiplying the numerator and denominator of the second term of the dividend by 2, those of the next by 2^2, &c.; and by dividing the terms of the first fraction in the divisor by 2, and multiplying those of the third and fourth respectively by 2 and 2^2, we obtain

$$\frac{y^5-3\times 2y^4+2^2y^3+3\times 2^3y^2-2^4y+3\times 2^5}{2^5} \,\Big|\, \frac{y^3+y^2-3\times 2y+2^2}{2^2}.$$

Hence, by performing the actual multiplications, and multiplying the dividend and divisor by 2^2, we get

$$\frac{y^5-6y^4+4y^3+24y^2-16y+96}{2^3} \,\Big|\, y^3+y^2-6y+4.$$

The rest of the operation is now reduced to the dividing of y^5-6y^4, &c., by y^3+y^2-6y+4, and taking one eighth of the quotient; and, as the first term of the divisor has the coefficient 1, the division will be performed in the manner already explained. We should thus obtain

$$\frac{y^2-7y+17}{2^3}-\frac{1}{2^3}\cdot\frac{39y^2-114y-28}{y^3+y^2-6y+4};$$

or,

$$\frac{2^2x^2-7\times 2x+17}{2^3}-\frac{1}{8}\cdot\frac{39\times 2^2x^2-114\times 2x-28}{2^3x^3+2^2x^2-6\times 2x+4},$$

by substituting for y its value, $2x$; whence, by contracting the first three terms, and by dividing the numerator and denominator of the fractional part by 4, we get

$$\frac{x^2}{2}-\frac{7x}{4}+\frac{17}{8}-\frac{1}{8}\cdot\frac{39x^2-57x-7}{2x^3+x^2-3x+1},$$

the same as before; and a comparison of the short process given above with the fuller one now explained, will show the virtual identity of the former with the latter. A general illustration would have been obtained

	Dividends.	Divisors.	Quotients.
16.	$4x^4-33x^2+8x-3$	$x+3$	$4x^3-12x^2+3x-1$
17.	x^4	$x+1$	$x^3-x^2+x-1+\frac{1}{x+1}$
18.	x^4	$x-1$	$x^3+x^2+x+1+\frac{1}{x-1}$
19.	x^4+3x^3-4	x^2+3	$x^2+3x-3-\frac{9x-5}{x^2+3}$
20.	$a^2-4b^2+c^2+2ac$	$a-2b+c$	$a+2b+c$

21. Divide $7x^4-26x^3+50x^2-74x+35$ by x^3-3x^2+5x-7
Ans. $7x-5$.

22. Divide $2x^4-3x^3y+2x^2y^2+y^4$ by $x-4y$.
Ans. $2x^3+5x^2y+22xy^2+88y^3+\frac{353y^4}{x-4y}$.

23. Divide x^5 by x^2+2x+1.
Ans. $x^3-2x^2+3x-4+\frac{5x+4}{x^2+2x+1}$.

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by using a, b, c, &c., as coefficients, instead of the particular ones in the present example. The principle, however, would be exactly the same; and what is here given will be more easily understood by beginners.

The following exercises are subjoined for the use of those who may wish to study this method more particularly.

Exer. 1. Divide $6x^{11}+7x^{10}v+4x^9v^2+16x^8v^3-5x^7v^4$ by $3x^4-x^3v+5x^2v^2$.
Ans. $2x^7+3x^6v-x^5v^2$.

Exer. 2. Divide $16x^4-8x^2z^2+z^4$ by $4x^2-4xz+z^2$.
Ans. $4x^2+4xz+z^2$.

Exer. 3. Divide $10x^6-11x^5-3x^4+20x^3+10x^2+2$ by $5x^3-3x^2+2x-2$.
Ans. $2x^3-x^2-2x+4+\frac{24x^2-12x+10}{5x^3-3x^2+2x-2}$.

Exer. 4. Divide $5x^5+2$ by $3x^2-2x+3$.
Ans. $\frac{5x^3}{3}+\frac{10x^2}{9}-\frac{25x}{27}-\frac{140}{81}-\frac{1}{81}\cdot\frac{55x-582}{3x^2-2x+3}$

It may be remarked, that, by dividing the dividend and divisor by the first coefficient of the divisor, we should be enabled to employ the method explained in the text. Thus, in Exer. 4., we should have simply to divide $\frac{5x^5}{3}+\frac{2}{3}$ by $x^2-\frac{2x}{3}+1$. In this way, however, the work would be embarrassed with fractions.

24. Divide x^5 by x^2-2x+1.

Ans. $x^3+2x^2+3x+4+\dfrac{5x-4}{x^2-2x+1}$.

25. Divide x^4-8x+7 by x^2-3x+2.

Ans. $x^2+3x+7+\dfrac{7x-7}{x^2-3x+2}$.

26. Divide $5x^4-6ax^3+2a^3x-a^4$ by x^2-ax+a^2.

Ans. $5x^2-ax-6a^2-\dfrac{3a^2x-5a^4}{x^2-ax+a^2}$.

27. Divide $x^6-3x^4v^2+3x^2v^4-v^6$ by $x^3-3x^2v+3xv^2-v^3$

Ans. $x^3+3x^2v+3xv^2+v^3$.

28. Divide $3x^6-37x^4+35x^3+7x^2+2$ by x^3+3x^2-4x-2.

Ans. $3x^3-9x^2+2x-1$.

29. Divide $9a^3b+9a^2bc-4ab^3+4b^3c-9abc^2-9bc^3$ by $3a-2b+3c$.

Ans. $3a^2b+2ab^2-2b^2c-3bc^2$.

30. Divide $a^3+3a^2b+3ab^2+2b^3+3b^2c+3bc^2+c^3$ by $a+2b+c$.

Ans. $a^2+ab+b^2+bc+c^2-ac$.

31. Divide $4x^5-x^3+4x$ by $2x^2+3x+2$. *Ans.* $2x^3-3x^2+2x$.

32. Divide $x-9x^9+8x^{10}$ by $1-2x+x^2$.

Ans. $x+2x^2+3x^3+4x^4+5x^5+6x^6+7x^7+8x^8$.

CHAPTER III.

MISCELLANEOUS PROPOSITIONS AND INVESTIGATIONS.

51. The methods of performing some of the more important elementary operations of algebra having now been established, it may be proper to apply them in the investigation of some principles of a miscellaneous character, several of which will often be found useful in subsequent inquiries. The following is one of much importance.

The form *of a quantity may be changed without changing its* value, *by first* performing *an operation which will alter the value, and then* indicating *the reverse operation.*

Thus, by adding $a+b$ to a, and then indicating the subtraction of the same, we get $a=2a+b-(a+b)$; by multiplying a by b, and indicating the division of the product by b, we find $a=\frac{ab}{b}$; and by taking the square root of 100, and indicating the squaring of that root, we have $100=10^2$. In like manner, if we divide $a^2x-abx+b^2x$ by x, and then indicate the multiplication of the quotient by x, we get $a^2x-abx+b^2x=(a^2-ab+b^2)x$. It is plain (§ 18.) that all the expressions thus found are identical equations.

52. The operations in the margin show, that, *if the difference of two quantities be added to their sum, the result is twice the greater; but, if the difference be taken from the sum, the remainder is twice the less.* In like manner, if we had used half the sum and half the difference, that is, by the note to § 44., $\frac{1}{2}a+\frac{1}{2}b$ and $\frac{1}{2}a-\frac{1}{2}b$, we should have found, that, *if half the difference be added to half the sum, the result is the greater; while, if half the difference be taken from half the sum, the remainder is the less.*

$$\begin{array}{rl} \text{To} & a+b \\ \text{add} & a-b \\ \hline \text{Sum}= & 2a \end{array} \qquad \begin{array}{rr} \text{From} & a+b \\ \text{take} & a-b \\ \hline \text{Rem.}= & 2b \end{array}$$

53. By multiplying $a+b$ by $a+b$, and $a-b$ by $a-b$, either in the common way, or by means of the coefficients, as in the margin, we find, that $(a+b)^2=a^2+2ab+b^2$, and $(a-b)^2=a^2-2ab+b^2$. By comparing these results with the binomials which respectively produce them, we see that the first and last terms in each are the squares of the two terms of the binomial, $b\times b$, in the one, and $-b\times -b$ in the other, being each equal to b^2: we see also, that in each the middle term is twice the product of the terms of the binomial; the one being $2ab$, or $2\times a\times b$, and the other $-2ab$, or $2\times a\times -b$: this being the way, in fact, in which, in the work, the middle terms originate. Hence, therefore, *the square of a binomial is equal to the squares of both its terms and twice their product.*[*]

$$\begin{array}{r} 1+1 \\ 1+1 \\ \hline 1+1 \\ 1+1 \\ \hline 1+2+1,\ \text{or} \\ a^2+2ab+b^2 \end{array} \qquad \begin{array}{r} 1-1 \\ 1-1 \\ \hline 1-1 \\ -1+1 \\ \hline 1-2+1,\ \text{or} \\ a^2-2ab+b^2. \end{array}$$

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[*] By multiplying $-a-b$ by $-a-b$, it would be found, that $(-a-b)^2=a^2+2ab+b^2$, which equally follows the rule given above. It will also be readily seen, that, by using the double sign, ±, the two expressions in the text may be combined in the one, $(a\pm b)^2=a^2\pm 2ab+b^2$. It ought to be observed, that whenever this double sign is used. one formula is obtained by taking, *throughout*, the upper sign, and another by taking the

54. By means of the foregoing theorem, we may often find the squares of compound quantities very easily, without going through with the formal multiplication. Thus, the square of $2x+3y$ is $4x^2+12xy+9y^2$; and that of $4a^2-3bc$ is $16a^4-24a^2bc+9b^2c^2$.

By the same means we may square the trinomial $a+b+c$ by writing it under any of the forms $(a+b)+c$, $a+(b+c)$, and $(a+c)+b$; and thus taking, at first, two terms as one. Thus, according to the first form, we have

$$(a+b+c)^2=(a+b)^2+2(a+b)c+c^2:$$

or, by squaring $a+b$ according to the last §, and performing the actual multiplication in the expression, $2(a+b)c$,

$$(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2.$$

The same would be obtained, merely with the terms differently arranged, from either of the other forms.

In like manner, we might find the square of $a+b+c+d$, by grouping the terms, two and two, or three and one, in different ways; thus, $(a+b)+(c+d)$, $(a+c)+(b+d)$, $(a+b+c)+d$, &c.; and thus we may proceed in numberless other instances.

55. If in the expression, $(a+b)^2=a^2+2ab+b^2$, we change every where b into $-b$[*], we get $(a-b)^2=a^2+2a\times-b+(-b)^2$; or, by actually performing the operations that are indicated, $(a-b)^2=a^2-2ab+b^2$, the same as the second expression in § 53. In a similar manner, when any formula whatever has been established, another will be obtained from it by changing the sign of one of the quantities concerned, wherever that quantity may occur. In all such cases, powers having even indices continue the same; but those which have odd indices, have their signs changed. This

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lower; and that, in making out separate formulas, the upper and the lower must not be taken in connexion with one another. Thus, in the last expression, if we take the upper sign in the first member, we must take the same in the second; and if we take the lower in the first, we must take the lower also in the second: as each of the expressions, $(a+b)^2=a^2-2ab+b^2$, and $(a-b)^2=a^2+2ab+b^2$, would be erroneous.

[*] Such changes as the present are often very useful in enabling us to derive formulas from others previously established; and, to make them legitimate, we have only to attend to the consequences which the change must produce in each particular instance. In the present case, and in all similar ones, we have merely to attend to the effect produced on the signs, a change of sign being the sole change made on the quantity: and it is plain, from considering the nature of the operations in addition, subtraction, multiplication, and division, that if we combine b and $-b$ with a quantity by means of any of these operations, the results will agree in every respect, except with regard to the signs.

follows from the fact, that, by the rule for the signs in multiplication, $(-a)^2=a^2$, $(-a)^4=a^4$, &c.; while $(-a)^3=-a^3$, $(-a)^5=-a^5$, &c.

In this way, by changing the sign of x, in Exer. 8. in multiplication, we get

$$(1+2x+3x^2+4x^3)\times(1-x)=1+x+x^2+x^3-4x^4,$$

the same that would be obtained by actual multiplication. So likewise, by changing the sign of y in the 7th exercise in division, we should find, that

$$\frac{x^4-9x^2y^2-12xy^3-4y^4}{x^2+3xy+2y^2}=x^2-3xy-2y^2;$$

and, by changing the sign of c in one of the expressions in the last §, we get

$$(a+b-c)^2=a^2+2ab+b^2-2ac-2bc+c^2.$$

56. By multiplying the members of $(a+b)^2=a^2+2ab+b^2$ by $a+b$, we get

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3, \text{ or, (§ 51.)}$$
$$(a+b)^3=a^3+3ab(a+b)+b^3;$$
$$\text{whence (§ 55.) } (a-b)^3=a^3-3ab(a-b)-b^3.$$

Hence, *the cube of a binomial is equal to the cubes of both its terms, and three times their product multiplied into the binomial.* Thus, for example, $(2a^2\pm3b^2)^3=8a^6\pm18a^2b^2(2a^2\pm3b^2)\pm27b^6$. By this means also we might cube trinomials or polynomials, after the manner pointed out in § 54.

57. If $a+b$ be multiplied by $a-b$, the product is a^2-b^2: that is, $(a+b)(a-b)=a^2-b^2$. Hence, *if the sum of two quantities be multiplied by their difference, the product is equal to the difference of their squares:* and, conversely, *the difference of two squares is equal to the product of the sum and difference of the roots of those squares.*

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\;\; -ab-b^2 \\ \hline a^2-b^2 \end{array}$$

$$\text{Hence, } (2a^2b+5ab^2)(2a^2b-5ab^2)=4a^4b^2-25a^2b^4;$$
$$(x^2+x+1)(x^2-x+1)=x^4+x^2+1;$$
$$(a+b+c)(a-b+c)=a^2+2ac+c^2-b^2; \text{ and}$$
$$(a+b-c)(a-b+c)=a^2-(b-c)^2=a^2-b^2+2bc-c^2.$$

In this last example the factors may be written thus, $a+(b-c)$ and $a-(b-c)$.

Again, $4x^2-9a^2$ may be resolved into the product of the factors, $2x+3a$ and $2x-3a$. In like manner, we should have $x^4-a^4=(x^2+a^2)(x^2-a^2)$, or, by a second and third resolution,

$$x^4-a^4=(x^2+a^2)(x+a)(x-a)$$
$$=(x^2+a^2)(x+a)(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a}).$$

The last resolution depends on the principle that x and a are the squares of $\sqrt{x}$ and $\sqrt{a}$; and on the same principle, the number of factors might be increased as much as we please, by introducing the sums and differences of the fourth roots, of the eighth roots, &c.

58. *If n be a whole positive number, x^n-a^n is divisible by $x-a$*; that is, in dividing it by $x-a$, there is no remainder; *and the quotient is*

$$x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots\ldots\ldots+a^{n-3}x^2+a^{n-2}x+a^{n-1}.*$$

To prove this, let us first take, for simplicity, the particular expression, x^3-a^3. Now (§ 51.) this may be put under the form $x^3-ax^2+ax^2-a^2x+a^2x-a^3$, as ax^2 and a^2x are each subtracted and added. In this, the first and second terms are divisible by x^2, the third and fourth by ax, and the remaining part by a^2. Hence, (§ 51.) $x^3-a^3=x^2(x-a)+ax(x-a)+a^2(x-a)$; the second member of which is divisible by $x-a$, and the result, x^2+ax+a^2, is of the form given above, when $n=3$.

The general proof proceeds in the same manner. Thus, x^n-a^n may be put under the form

$$x^n-ax^{n-1}+ax^{n-1}-a^2x^{n-2}+a^2x^{n-2}-\ldots-a^{n-1}x+a^{n-1}x-a^n.$$

In this (§ 51.) the first two terms are equivalent to $x^{n-1}(x-a)$; the last pair to $a^{n-1}(x-a)$; the second pair to $ax^{n-2}(x-a)$; the pair before the last pair to $a^{n-2}x(x-a)$; and so on. Hence, dividing by $x-a$, we get, without remainder,

$$\frac{x^n-a^n}{x-a}=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots\ldots+a^{n-3}x^2+a^{n-2}x+a^{n-1}.$$

59. From what was established in the last §, we have, conversely, by multiplying by $x-a$,

$$(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots\ldots+a^{n-2}x+a^{n-1})(x-a)=x^n-a^n;$$

---

* In this, and in all similar expressions, the dots are used to denote terms which are not written, but which may be introduced at pleasure, when the law of formation is known. In the present case, in each term, the index of x is less, and that of a greater, by unity, than the corresponding index in the preceding term. It may be remarked, that the word *divisible*, as used above, and in many similar instances, is employed to denote, that when one quantity is divided by another, there is no remainder.

or, by putting $n-1=m$, and consequently $n-2=m-1$, &c.

$$(x^m+ax^{m-1}+a^2x^{m-2}+\ldots\ldots+a^{m-1}x+a^m)(x-a)=x^{m+1}-a^{m+1}.$$

It may be observed, that in giving particular values to m or n, the series will terminate, according to the original assumption in the investigation, when the index of x becomes 0. Thus, in the last expression, if $m=1$, we have simply $(x+a)(x-a)=x^2-a^2$: and it thus appears, that the principle established in § 57. is only a particular case of the general principle investigated in § 58.

60. If the sign of a be changed, $x-a$ becomes $x+a$; and, since (§ 55.) the even powers of $-a$ and a are the same, while their odd powers have opposite signs; it follows, that, this change of sign being made, x^n-a^n continues the same when n is even, but becomes x^n+a^n when n is odd. Hence, when n is odd x^n+a^n is divisible by $x+a$; and, when it is even, x^n-a^n is divisible by $x+a$ as well as (§ 58.) by $x-a$: and, therefore, since, if m be a whole number, $2m$ denotes an even number, and $2m+1$ (or $2m-1$) an odd one, $x^{2m}-a^{2m}$ is divisible by either $x+a$ or $x-a$; and $x^{2m+1}+a^{2m+1}$ by $x+a$. It follows, accordingly, from § 58., that

$$\frac{x^{2n}-a^{2n}}{x-a}=x^{2n-1}+ax^{2n-2}+a^2x^{2n-3}+\ldots\ldots+a^{2n-1};$$

and hence, by changing the sign of a,

$$\frac{x^{2n}-a^{2n}}{x+a}=x^{2n-1}-ax^{2n-2}+a^2x^{2n-3}-\ldots\ldots-a^{2n-1}.$$

Also,

$$\frac{x^{2n+1}+a^{2n+1}}{x+a}=x^{2n}-ax^{2n-1}+a^2x^{2n-2}-\ldots\ldots+a^{2n}.$$

It would be easily shown also, that

$$\frac{x^{2n}-a^{2n}}{x^2-a^2}=x^{2n-2}+a^2x^{2n-4}+a^4x^{2n-6}+\ldots\ldots+a^{2n-2}.*$$

61. By means of these principles, we can often obtain the results of operations in division without performing the operations in the usual way. Thus,

$$\frac{x^7-a^7}{x-a}=x^6+ax^5+a^2x^4+a^3x^3+a^4x^2+a^5x+a^6;$$

$$\frac{x^7+a^7}{x+a}=x^6-ax^5+a^2x^4-a^3x^3+a^4x^2-a^5x+a^6;$$

---

* This would be shown by changing x into x^2, and a into a^2 in the formula at the end of § 58.; and it might be proved in other modes.

$$\frac{x^6-a^6}{x-a}=x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5;$$

$$\frac{x^6-a^6}{x+a}=x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5;$$

$$\frac{x^3\pm a^3}{x\pm a}=x^2\mp ax+a^2;$$

$$\frac{32x^5-1}{2x-1},\text{ or }\frac{(2x)^5-1}{2x-1}=16x^4+8x^3+4x^2+2x+1.$$

If, again, it were required to divide $3x^4-5x^3+5x-3$ by $x-1$, we might put the dividend under the form, $3(x^4-1)-5x(x^2-1)$. Then, dividing the factors in the vinculums by $x-1$, we should get for quotient $3(x^3+x^2+x+1)-5x(x+1)$, or, by easy reductions, $3x^3-2x^2-2x+3$; and this might be divided in a similar manner by $x+1$.

62. We have now had some of the most interesting cases in which quantities are exactly divisible by compound divisors. In numberless other instances, there will always be a remainder, however far the work be carried; and the quotient might be made to consist of an infinite number of monomials. In such cases we may terminate the work at any point we please, by annexing to the part of the quotient previously found a fraction having the remainder as numerator, and the divisor as denominator, as was shown in division; or we may carry out the process, till we see the law of continuation, by which the subsequent terms may be derived from those already found.

As a simple instance, let 1 be divided by $1-r$. Here the first term of the quotient is 1 with the remainder r; the first two terms are $1+r$ with the remainder r^2; the first three, $1+r+r^2$, with r^3: and it is plain, that we may proceed thus, as far as we please.

$$\begin{array}{l|l}
1 & 1-r \\
\underline{1-r} & 1+r+r^2+r^3+,\ \&c. \\
\quad r & \\
\quad \underline{r-r^2} & \\
\qquad r^2 & \\
\qquad \underline{r^2-r^3} & \\
\qquad\quad r^3 & \\
\qquad\quad \underline{r^3-r^4} & \\
\qquad\qquad r^4,\ \&c. &
\end{array}$$

Hence we have successively,

$$\frac{1}{-r}=1+\frac{r}{1-r}=1+r+\frac{r^2}{1-r}=1+r+r^2+\frac{r^3}{1-r},\ \&c.$$

In the first of these quotients, there is one term exclusive of the remainder, in the second there are two, in the third three, and so on; and in each of them the highest index of r is *one* less than the number of terms. Thus, in the second term we have r^1, in the third r^2, &c. We see also, that the index of r in the fractional part is always the same as the number of terms preceding the fraction. Hence, taking n to denote any number of terms, we have the general expression,

$$\frac{1}{1-r}=1+r+r^2+r^3+\ldots\ldots\ldots+r^{n-1}+\frac{r^n}{1-r}.$$

In this, r^{n-1} is called the nth *term*, or the *general term*, because it will give any particular term if n be taken of the proper value. Thus, by taking $n=4$, we find the fourth term to be r^3, with the remainder r^4, as we saw in the operation: and if we take $n=50$, we find the fiftieth term to be r^{49}, with the remainder r^{50}.

63. If we change the sign of r in the formula found in the last §, we get

$$\frac{1}{1+r}=1-r+r^2-r^3+\ \ldots\ldots\ \pm r^{n-1}\mp\frac{r^n}{1+r}.$$

Here, the signs will be alternately + and —, the even powers of $-r$ being (§ 55.) positive, and the odd ones negative; and the general term and the fractional one will be sometimes positive and sometimes negative. These will likewise have opposite signs, because, as their indices differ by unity, if the one be even, the other will be odd. As examples, by taking n successively equal to 25 and 30, we find the twenty-fifth term to be r^{24}, with the remainder $-r^{25}$; and the thirtieth to be $-r^{29}$, with the remainder r^{30}. Since $-r$ is the same as $-1r$, the last formula may be thus expressed:

$$\frac{1}{1+r}=1-r+r^2-r^3+\ldots\ldots\ldots+(-1)^{n-1}r^{n-1}+\frac{(-1)^n r^n}{1+r}.$$

64. By changing r into $\frac{b}{a}$ in the formula found in § 62., and in the last in § 63., by multiplying the numerators and denominators in some terms of the results by a, and by making some other obvious modifications, we get

$$\frac{a}{a-b}=1+\frac{b}{a}+\frac{b^2}{a^2}+\frac{b^3}{a^3}+\ \ldots\ldots\ +\frac{b^{n-1}}{a^{n-1}}+\frac{b^n}{a^{n-1}(a-b)},$$

and

$$\frac{a}{a+b}=1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\ \ldots\ldots\ +\frac{(-1)^{n-1}b^{n-1}}{a^{n-1}}+\frac{(-1)^n b^n}{a^{n-1}(a+b)}.$$

65. By dividing 1 by $r+1$, as below, we get,

$$\frac{1}{r+1}=r^{-1}-r^{-2}+r^{-3}-r^{-4}+\ldots\ldots+(-1)^{n-1}r^{-n},$$
$$=\frac{1}{r}-\frac{1}{r^2}+\frac{1}{r^3}-\frac{1}{r^4}+\ldots\ldots\ldots+\frac{(-1)^{n-1}}{r^n}.^{*}$$

This is the quotient obtained, in descending powers of r, in dividing 1 by $r+1$; and in § 63. there is another expression for the same in ascending powers.

$$\begin{array}{l|l}
1 & r+1 \\
\underline{1+r^{-1}} & r^{-1}-r^{-2}+r^{-3}-r^{-4}+ \text{ \&c.} \\
-r^{-1} & \\
\underline{-r^{-1}-r^{-2}} & \\
\qquad r^{-2} & \\
\qquad \underline{r^{-2}+r^{-3}} & \\
\qquad\quad -r^{-3}, \text{ \&c.} &
\end{array}$$

66. We saw, in the last §, that the division of 1 by $r+1$ was facilitated by employing negative indices, all trouble in the management of fractions being avoided in the course of the operation; and the same method may be followed with advantage in many other cases. The operation in the margin, in which x^2 is divided by $x-3a$, affords another instance. By examining the quotient and the mode of its formation in this example, it will be readily seen, that the general term is $3^{n-1}a^{n-1}x^{-n+2}$, with the remainder, $3^na^nx^{-n+2}$; and the powers of x which have negative indices may be expressed in the ordinary way, in the manner pointed out in § 46.

$$\begin{array}{l|l}
x^2 & x-3a \\
\underline{x^2-3ax} & x+3a+3^2a^2x^{-1}+3^3a^3x^{-2}+\text{ \&c.} \\
\quad 3ax & \\
\quad \underline{3ax-3^2a^2} & \\
\qquad 3^2a^2 & \\
\qquad \underline{3^2a^2-3^3a^3x^{-1}} & \\
\qquad\quad 3^3a^3x^{-1} & \\
\qquad\quad \underline{3^3a^3x^{-1}-3^4a^4x^{-2}} & \\
\qquad\qquad 3^4a^4x^{-2}, \text{ \&c.} &
\end{array}$$

Exercises.

1, 2, 3, 4, and 5. Show, by § 53.,

that $(x^2+x+1)^2=x^4+2x^3+3x^2+2x+1$;

that $(x^3-x^2+x-1)^2=x^6-2x^5+3x^4-4x^3+3x^2-2x+1$;

that $(x\pm x^{-1})^2=x^2\pm 2+x^{-2}$;

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* This might also be obtained from the last formula in § 63., by changing r into r^{-1}, and dividing both members of the result by r.

that $(x^m \pm x^n)^2 = x^{2m} \pm 2x^{m+n} + x^{2n}$;
and that $(x^{-m} \pm x^{-n})^2 = x^{-2m} \pm 2x^{-m-n} + x^{-2n}$.

6, and 7. Show, by § 55., from Exam. 9., in Multiplication,

$$(x^4 + 3ax^3 + 4a^2x^2) \times (x^3 - 3ax^2 - a^2x)$$
$$= x^7 - 6a^2x^5 - 15a^3x^4 - 4a^4x^3;$$

and, from Exer. 9. in Division, that, if $x^2 + a^2$ be divided by $x + a$, the quotient is $x - a + \dfrac{2a^2}{x+a}$.

8, 9, and 10. Prove, from § 56.,

that $(2a \pm 5b)^3 = 8a^3 \pm 30ab(2a \pm 5b) \pm 125b^3$
$= 8a^3 \pm 60a^2b + 150ab^2 \pm 125b^3$;
that $(x - x^{-2})^3 = x^3 - 3 + 3x^{-3} - x^{-6}$;
and that $(1 - x + x^2)^3 = 1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6$.

11, 12, 13, 14, 15, 16, and 17. From § 57., prove

that $(4ax + 3x^2)(4ax - 3x^2) = 16a^2x^2 - 9x^4$;
that $(1 + x + x^2)(1 - x + x^2) = 1 + x^2 + x^4$;
and that $(x + x^{-1})(x - x^{-1}) = x^2 - x^{-2}$.

Prove also, conversely,

that $x^2 - 4a^2 = (x + 2a)(x - 2a)$;
that $x^2 + 2ax + a^2 - b^2 = (x + a + b)(x + a - b)$;
that $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$;
and that $a^2 - b^2 - 2bc - c^2$, or $a^2 - (b + c)^2 = (a + b + c)(a - b - c)$.

18, 19, 20, 21, and 22. From §§ 58, 59, and 60., prove that, if $x^5 + 1$ be divided by $x + 1$, the quotient is $x^4 - x^3 + x^2 - x + 1$; and that if $81a^4 - 16b^4$ be divided by $3a - 2b$, the quotient is $27a^3 + 18a^2b + 12ab^2 + 8b^3$.

Prove also,

that $(32a^5 - 48a^4y + 72a^3y^2 - 108a^2y^3 + 162ay^4 - 243y^5)(2a + 3y)$
$= 64a^6 - 729y^6$;

that $(x + 1 + x^{-1})(x - 1) = x^2 - x^{-1}$; and that if $5x^4 - x^3 + x - 5$ be divided by $x - 1$, the quotient is $5x^3 + 4x^2 + 4x + 5$.

23. Show, that if 1 be divided by $1 + r^2$, the quotient is either

$$1-r^2+r^4-r^6+\ldots\ldots\ldots+(-1)^{n-1}r^{2n-2},$$
$$\text{or } r^{-2}-r^{-4}+r^{-6}-r^{-8}+\ldots\ldots\ldots+(-1)^{n-1}r^{-2n};$$

and that if 1 be divided by $1-2x+x^2$, the quotient is

$$1+2x+3x^2+4x^3+\ldots\ldots+nx^{n-1}+\frac{(n+1)x^n-nx^{n+1}}{1-2x+x^2}.$$

24. Show that $x^7+x^6+x^5+x^4+x^3+x^2+x+1$ may be put successively under the forms,

$$(x^4+1)(x^3+x^2+x+1) \text{ and } (x^4+1)(x^2+1)(x+1).$$

25. Prove that $x^7-x^6-x^5+x^4-x^3+x^2+x-1$

$$=(x^4-1)(x^3-x^2-x+1)=(x^2+1)(x+1)^2(x-1)^3.$$

26. By dividing 1 by $1-r+r^2$, and by changing r into $-r$ in the result, find the developements of two different fractions in infinite series.

Ans. $\dfrac{1}{1-r+r^2}=1+r-r^3-r^4+r^6+r^7-r^9-r^{10}+$ &c.;

and $\dfrac{1}{1+r+r^2}=1-r+r^3-r^4+r^6-r^7+r^9-r^{10}+$ &c. *

27. Show that the series in the last question may be written thus: $(1+r)s_1$, and $(1-r)s_2$; where $s_1=1-r^3+r^6-r^9+$ &c., and $s_2=1+r^3+r^6+r^9+$ &c.

28. Prove that $(a+b)^2-c^2+(a+c)^2-b^2+(b+c)^2-a^2=(a+b+c)^2$.†

29. Divide x^2+2a^2 by x^2+a^2, using negative indices; and determine the nth term, and the 120th term, of the quotient.

Ans. $1+a^2x^{-2}-a^4x^{-4}+a^6x^{-6}-\ldots+(-1)^n a^{2n-2}x^{-2n+2}$;
and the 120th term $=a^{238}x^{-238}$.

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* In the first of these series, if the first and second terms be multiplied by $-r^3$, the products will be the third and fourth; if these be multiplied by the same, the products will be the fifth and sixth; and so on; and thus the law of continuation is evident. In the second the multiplier is r^3. It will be seen also, that all the powers of r are wanting, whose indices are of the form $3n+2$, (or $3n-1$), n being any number in the series, 0, 1, 2, 3, &c.

† This and similar questions, such as Exer. 30. and 31., may be solved *synthetically*, by performing the operations indicated in both members, and showing that the results are equal. For the improving and training of his mind, however, the student should endeavour to discover *analytical* solutions; that is, he should take one of the members, suppose the first, and, having performed the operations indicated in it, he should try to reduce the result so that it may become equal to the other member.

30. Show that $(x+y)^3+(x-y)^3=2x(x^2+3y^2)$,
and $(x+y)^3-(x-y)^3=2y(y^2+3x^2)$.

31. Prove that
$(a+b)(a+c)(b+c)=a^2(b+c)+b^2(a+c)+c^2(a+b)+2abc$.

32. From the second, third, fourth, and fifth powers of $x+h$, take the same powers of x, and divide the remainders by h.

Ans. $2x+h$, $3x^2+3xh+h^2$, $4x^3+6x^2h+4xh^2+h^3$, and $5x^4+10x^3h+10x^2h^2+5xh^3+h^4$.

33. Prove that the square of a^2+b^2 is equal to the sum of the squares of a^2-b^2 and $2ab$.[*]

34. Required the continual product of $a+b-2c$, $a-2b+c$, and $-2a+b+c$.

Ans. $3\{a^2(b+c)+b^2(a+c)+c^2(a+b)\}-2(a^3+b^3+c^3)-12abc$.

CHAPTER IV.

FRACTIONS.[†]

67. When the division of one quantity by another is merely indicated, the expression is called a *fraction;* and, by means of terms borrowed from arithmetic, the divisor is called the *denomi-*

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[*] Hence, (Euc. I. 48.) if a and b be any two unequal whole numbers, a^2+b^2 will be the hypotenuse, and a^2-b^2 and $2ab$ the legs of a right-angled triangle, having its sides whole numbers. Thus, if we take $a=2$, and $b=1$, we find the sides to be 3, 4, and 5; while, if we assume 4 and 3, we get 7, 24, and 25.

[†] As a great part of the management of algebraic fractions is almost identical with that of arithmetical ones, and as the student is supposed to be acquainted with the modes of treating the latter before commencing algebra, operations have already been performed in some instances on fractional quantities. The following Section, however, is written without reference to any knowledge of the subject, which the student may previously possess.

The management of fractions consists partly in certain modifications of their forms, which, without changing their values, prepare them for ulterior operations, and which constitute *reduction* of fractions; and partly in the performance of such operations on fractional quantities, as are performed on others, such as their *addition, subtraction, multiplication,* &c.

nator of the fraction, and the dividend its *numerator*. Thus, $\frac{\mathrm{N}}{\mathrm{D}}$, or, as it may be written, ND^{-1}, is a fraction; and D is its denominator, and N its numerator. The numerator and denominator of a fraction are often called, for brevity, its *terms*.

The following are three fundamental principles, on which much of the theory of fractions depends.

68. ***If the terms of a fraction be either both multiplied, or both divided, by the same quantity, the fraction so obtained is equal to the original one.***

69. *To multiply a fraction by a quantity,* either multiply its numerator, or divide its denominator, by that quantity.

70. *To divide a fraction by a quantity,* either divide its numerator, or multiply its denominator, by that quantity.

71. These principles are proved in the margin; where, for ease of reference, the successive expressions are marked (1.), (2.), (3.), &c., and where F denotes the original fraction or quotient, $\frac{\mathrm{N}}{\mathrm{D}}$. Then, since, by the the nature of division, a quotient, when there is no remainder, is such a quantity, that, if it be multiplied by the divisor, the product is the dividend, we get (2.) by multiplying the members of (1.) by D. From this (Ax. 3. p. 9.) we obtain (3.) by multiplying both members by the same quantity m. We then divide (Ax. 4. p. 9.) both members of this by mD to get (4.); and (5.) is obtained by equalling the values of F in (1.) and (4.). This formula proves § 68.; since, if the terms of the first member be multiplied by m, the result is the second; while, if the terms of the second be divided by m, the result is the first; and the two members have been proved to be equal.

$$\mathrm{F}=\frac{\mathrm{N}}{\mathrm{D}} \ldots\ldots (1.)$$

$$\mathrm{DF}=\mathrm{N} \ldots\ldots (2.)$$

$$m\mathrm{DF}=m\mathrm{N} \ldots (3.)$$

$$\mathrm{F}=\frac{m\mathrm{N}}{m\mathrm{D}} \ldots (4.)$$

$$\frac{\mathrm{N}}{\mathrm{D}}=\frac{m\mathrm{N}}{m\mathrm{D}} \ldots (5.)$$

$$\left.\begin{array}{r} m\mathrm{F}=\frac{m\mathrm{N}}{\mathrm{D}} \\ \text{or, } m\frac{\mathrm{N}}{\mathrm{D}}=\frac{m\mathrm{N}}{\mathrm{D}} \end{array}\right\} . (6.)$$

$$\frac{\mathrm{F}}{m}=\frac{\mathrm{N}}{m\mathrm{D}} \ldots (7.)$$

Formula (6.) is derived (Ax. 4. p. 9.) from (3.) by dividing by D. By comparing this with formula (1.), we see that m times F is found by multiplying N, the numerator, by m, and retaining the denominator; while, by comparing it with (4.), we perceive that it is obtained by dividing the denominator of the second member by m, and retaining the numerator; and thus we have a proof of § 69.

Lastly, by dividing the members of (2.) by mD, and modifying the result by § 68. we get (Ax. 4. p. 9.) formula (7.). Now, the first member of this is the quotient found by dividing the fraction F by m; and its value, the second member, is obtained either by dividing the numerator of the second member of (4.), or by multiplying the denominator of the second member of (1.), by m; and these members being each equal to F, we have thus a proof of § 70.

72. If, according to § 70., we divide $\frac{N}{D}$ by N, and indicate the multiplication of the result by N, we get $\frac{N}{D}=N\frac{1}{D}$. Hence, it appears, that, *in case of any fraction, if a unit or integer be divided into as many parts as there are units in the denominator, the fraction expresses as many of these parts as there are units in the numerator.*[*]

73. *Fractions having different denominators may be reduced to equivalent ones having a common denominator,* by multiplying each numerator by all the denominators except its own, to find the numerators, and by taking the product of all the denominators as the common denominator.

Thus, $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are equivalent to $\frac{adf}{bdf}$, $\frac{bcf}{bdf}$, and $\frac{bde}{bdf}$. The reason is plain from § 68., since the terms of the first fraction are simply multiplied by d and f, all the denominators except its own; and in like manner, the terms of the second fraction are multiplied by bf, and those of the third by bd.

By a similar reduction, we find, that $\frac{a}{a+x}$ and $\frac{a}{a-x}$ are equivalent to $\frac{a^2-ax}{a^2-x^2}$ and $\frac{a^2+ax}{a^2-x^2}$.

74. *When some of the denominators have one or more common factors,* the results may be exhibited in simpler forms than those which would be found by the foregoing method. To accomplish

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[*] This view of the nature of a fraction is that which is ordinarily adopted in arithmetic, instead of the one given in § 67. Thus, for instance, in reference to a day, the meaning of the fraction $\frac{3}{4}$, according to the common mode of reading it, is, that a day is divided into four equal parts, and that the fraction expresses three of those parts. The two views are evidently identical, each being derivable from the other.

this, omit all the factors common to two or more denominators, except one of each, and take the product of all the remaining quantities for the common denominator. Then, divide this successively by the given denominators, and multiply the several quotients by the respective numerators, to find the required numerators.

Thus, if the fractions be $\frac{a}{bc}$, $\frac{d}{ce}$, $\frac{f}{bg}$, by omitting c in the second denominator and b in the third, we have remaining bc, e, and g; the product of which, $bceg$, is the common denominator. Dividing this by bc, and multiplying the quotient by a, we get aeg, the first numerator; and in a similar manner we find the two remaining numerators to be bdg and cef.

In like manner, if we had $\frac{a}{(a+x)^2}$, $\frac{b}{a^2-x^2}$, and $\frac{c}{(a-x)^2}$, since the denominators may be written $(a+x)(a+x)$, $(a+x)(a-x)$, and $(a-x)(a-x)$, we omit one factor, $a+x$, in the second, and one, $a-x$, in the third; and, multiplying the remaining factors together, we get for the common denominator, $(a+x)(a+x)(a-x)(a-x)$, which may be put under any of the forms $(a+x)^2(a-x)^2$, $(a+x)(a-x)(a+x)(a-x)$, $(a^2-x^2)^2$, &c. Then, dividing this by $(a+x)^2$, and multiplying the quotient by a; dividing again by a^2-x^2, and multiplying by b; and lastly, by dividing by $(a-x)^2$, and multiplying by c, we find the numerators to be $a(a-x)^2$, $b(a^2-x^2)$, and $c(a+x)^2$. The reason of the process is evident from § 68.

Exercises. Reduce the following sets of fractions to equivalent sets, having common denominators.

1. $\frac{1}{x}$, $\frac{2}{y}$, and $\frac{3}{z}$. *Ans.* $\frac{yz}{xyz}$, $\frac{2xz}{xyz}$, $\frac{3xy}{xyz}$.

2. $\frac{a}{b}$ and $\frac{b}{a}$. *Ans.* $\frac{a^2}{ab}$ and $\frac{b^2}{ab}$.

3. $\frac{a}{x^2}$ and $\frac{b}{y}$. *Ans.* $\frac{ay}{x^2y}$ and $\frac{bx^2}{x^2y}$.

4. $\frac{a+x}{b}$ and $\frac{a-x}{c}$. *Ans.* $\frac{ac+cx}{bc}$ and $\frac{ab-bx}{bc}$.

5. $\frac{1}{ab}$, $\frac{1}{bc}$, and $\frac{1}{a^2}$. *Ans.* $\frac{ac}{a^2bc}$, $\frac{a^2}{a^2bc}$, and $\frac{bc}{a^2bc}$.

6. $\frac{1}{a^2b^3}$ and $\frac{1}{a^3b^2}$. *Ans.* $\frac{a}{a^3b^3}$ and $\frac{b}{a^3b^3}$.

7. $\frac{a}{x^3+x^2+x+1}$ and $\frac{1}{x-1}$. *Ans.* $\frac{ax-a}{x^4-1}$ and $\frac{x^3+x^2+x+1}{x^4-1}$.

8. $\frac{x-a}{x^2-ax+a^2}$ and $\frac{1}{x+a}$. *Ans.* $\frac{x^2-a^2}{x^3+a^3}$ and $\frac{x^2-ax+a^2}{x^3+a^3}$.

9. $\frac{1}{2ab}, \frac{2}{3bc}, \frac{3}{4cd}, \frac{4}{5de}, \frac{5}{6ef}$.

Ans. $\frac{30cdef}{60abcdef}, \frac{40adef}{60abcdef}, \frac{45abef}{60abcdef}, \frac{48abcf}{60abcdef}$, and $\frac{50abcd}{60abcdef}$.

75. *To convert a mixed quantity* (that is, a quantity which is partly integral and partly fractional) *into a fraction;* multiply the integral part by the denominator: to the product add the numerator, if the fraction be preceded by +; otherwise, subtract it: lastly, below the result, write the denominator.* The reason is plain from § 51.

Thus, $x+a+\frac{a^2}{x-a}=\frac{x^2-a^2+a^2}{x-a}=\frac{x^2}{x-a}$;

and $x+a-\frac{x^2+a^2}{x+a}=\frac{x^2+2ax+a^2-(x^2+a^2)}{x+a}=\frac{2ax}{x+a}$.

An integral quantity is reduced to a fractional form, by multiplying it by any quantity, and taking that quantity as denominator, and the product as numerator. Thus, $x=\frac{ax}{a}$.

Exercises. Reduce the following quantities to fractions.

10. $x^2+ax+a^2+\frac{a^3}{x-a}$. *Ans.* $\frac{x^3}{x-a}$.

11. $2x-4a+\frac{7a^2}{x+2a}$. *Ans.* $\frac{2x^2-a^2}{x+2a}$.

---

* The term *improper fraction*, which is often applied to the result obtained by this rule, ought to be confined to arithmetic; as, in general, there is neither propriety nor use in calling algebraic fractions either proper or improper.

The converse problem, *the reducing of a fraction to a whole or mixed quantity*, is wrought by simply performing the division that is indicated.

12. $3a-4x+\frac{25ax-12x^2}{4a-3x}$. *Ans.* $\frac{12a^2}{4a-3x}$.

13. $1-\frac{x-a}{x+a}$. *Ans.* $\frac{2a}{x+a}$.

76. If the terms of a fraction, without becoming themselves fractional, can be divided by a common factor or measure, the result, which (§ 68.) is equivalent to the original fraction, is said to be in *lower terms:* and if the divisor so employed be the largest number possible, or the factor of the highest order, or, as it is generally called, *the greatest common measure or divisor*, the resulting fraction is said to be in its *lowest terms.* In establishing the method of finding the greatest common divisors of quantities, and thus of preparing for the reduction of fractions to their lowest terms, the principles given in the next § and the following are necessary.

77. If a quantity be a measure of two others, it is a measure of their sum and difference; and it is also a measure of either of them multiplied by any quantity. Thus, ma and mb have m as a common measure; and their sum and difference, $ma+mb$ and $ma-mb$, have the same common measure; m being contained, without remainder $a+b$ times in the one, and $a-b$ times in the other. Multiplying also ma by n, we get nma, which is still divisible by m.

78. If two quantities, mA and mB, have the quantity m as their greatest common measure, so that A and B are *prime* to each other, that is, have no common divisor except unity; and if one of them, mA, be multiplied by a quantity n which has no factor in common with B, m is likewise the greatest common measure of mB and the product nmA. For, dividing the two latter by m, which evidently measures them, we get B and nA; and, neither n nor A having any measure in common with B, nA and B have no common measure greater than unity. The greatest divisor, therefore, of mB and nmA is m. If, instead of multiplying, we had divided, mA by n, it would have been shown in the same manner that the quotient and mB have no common divisor greater than m.*

---

* Thus, $20=4\times 5$, and $24=4\times 6=4\times 2\times 3$; of which 4 is evidently the greatest common measure. Now, if we multiply 20 by 5, 7, 11, or any other number which is not 2 or 3, or a multiple or power of 2 or 3. or the product of any powers of these two numbers, the greatest common

79. When the terms of a fraction have a simple quantity as factor, it can be readily discovered by inspection. Thus, we per-

measure of the result and of 24 will still be 4. Thus, multiplying by 5, we have $100=5\times 20=4\times 5\times 5$, which has no factor in common with $24=4\times 2\times 3$ except 4. If, on the contrary, we multiply by 2, we get $40=2\times 4\times 5=8\times 5$, the greatest common measure of which and of 24 is 8.

In like manner, if instead of A and B we take cd and ef, which are prime to one another, so that c and d have no factors in common with e and f, we have mcd and mef, of which m is the greatest common factor: and it is plain, that if we multiply mcd by any quantity except e and f, and their powers, or the products of those powers, m will still be the greatest common divisor of the product and of mef. Thus, $mcdg$ and mef have m as their greatest common measure.

A strict proof, that, if neither n nor A have a measure in common with B, nA and B have no common measure greater than unity, may be as follows. Suppose m to be a factor of nA and yet to be prime to A: then m must be a factor of n. For, if we perform the process for finding the greatest common measure of m and A, we shall at length arrive at the remainder 1; and the work will stand as follows, $q_1, q_2, \ldots, q_n$, and $r_1, r_2, \ldots, 1$, denoting the respective quotients and remainders.

$$\begin{array}{l}
m)\,\underline{A}\,(q_1 \\
\quad r_1)\,\underline{m}\,(q_2 \\
\qquad r_2)\,\underline{r_1}\,(q_3 \\
\qquad\quad r_3 \\
\qquad\quad \ldots\ldots \\
\qquad\quad \ldots\ldots \\
\qquad\qquad r_{n-1})\,\underline{r_{n-2}}\,(q_n \\
\qquad\qquad\qquad r_n=1.
\end{array}$$

Hence,
$$A=mq_1+r_1,$$
$$m=r_1q_2+r_2,$$
$$r_1=r_2q_3+r_3,$$
$$r_2=r_3q_4+r_4,$$
&c.

Multiply the several quantities in the second column by n, and divide the results by m: then,

$$\frac{nA}{m}=nq_1+\frac{nr_1}{m}; \qquad \frac{nr_1}{m}=\frac{nr_2}{m}\cdot q_3+\frac{nr_3}{m}$$

$$n=\frac{nr_1}{m}\cdot q_2+\frac{nr_2}{m}; \qquad \frac{nr_2}{m}=\frac{nr_3}{m}\cdot q_4+\frac{nr_4}{m}, \text{ \&c.}$$

Now, in the first of these equations, $\frac{nA}{m}$ and nq, being whole numbers, $\frac{nr_1}{m}$ must be a whole number also, or the second member would be fractional. In the second equation, n is a whole number; and, by what has just been proved, the first term of the second member is a whole number: hence, the last term, $\frac{nr_2}{m}$ must likewise be a whole number: and it would be shown, in a similar manner, that the last term in each of the equations is an integer. Now, by continuing the process, we should arrive at the ceive at once, that the terms of the fraction $\frac{6ax^2-9a^2x}{3ax+12a^2}$, are divisible by $3a$, and that it is therefore equivalent to $\frac{2x^2-3ax}{x+4a}$.

In like manner, we readily see that

$$\frac{a^3b^2c}{a^2b^2c^2}=\frac{a}{c}, \text{ and that } \frac{2x^2y-4xy^2}{6x^2y^2}=\frac{x-2y}{3xy}.$$

Should there be large numerical coefficients, their greatest common measure may be found in the manner shown, Arithmetic, pages 81, 82, and 83.

80. When the common measure of two quantities is compound, it may be found in many cases, by means of the principles established in §§ 53, 56, 57, 58, &c.

Thus, we readily see that $x-a$ is a common divisor of x^3-a^3 and x^2-a^2; and $x+a$ of x^5+a^5 and x^3+a^3. In like manner, since (§ 51.) the fraction $\frac{3ax^4-4a^2x^3+4a^3x^2-3a^4x}{4x^4-8ax^3+4a^2x^2}$, may be put under the form $\frac{3ax(x^3-a^3)-4a^2x^2(x-a)}{4x^2(x^2-2ax+a^2)}$, we have $x-a$ as the common factor; and by dividing both terms by it and by x, we get, after some easy modifications, $\frac{3ax^2-a^2x+3a^3}{4x^2-4ax}$.

81. In general, however, *when there is a compound factor common to two quantities*, it cannot be found by inspection. In such cases, it will be obtained by the following rule: —

(1.) Divide one of the quantities by the other, using as divisor the one of the lower degree, if their degrees be different. (2.) If there be a remainder, divide the last divisor by it. (3.) Continue the operation in this manner, always dividing the divisor last employed by the last remainder, till nothing remains; and the divisor which leaves no remainder is the greatest common measure.

For the purpose of simplifying the operation, it ought to be

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remainder 1, and the corresponding last term would be $\frac{n}{m}$, which must, therefore, be a whole number, so that m must be a measure of n.

Hence, if m were a factor of nA and B, without being a factor of A, it must be also a factor of n: and therefore, if neither n nor A have a measure in common with B, nA and B can have no factor in common.

carefully observed, that when all the terms of either of the original quantities, or of any of the remainders that by the rule are to be used as divisors, can be divided by a common factor *, the division should be performed, and the result used instead of the quantity. In addition to this, every dividend should be multiplied, when necessary, by such a simple quantity as shall prevent the quotient from being fractional.

Exam. 1. Reduce $\frac{2x^4-x^3-10x^2-11x+8}{2x^3-3x^2-9x+5}$ to its lowest terms.

$$\begin{array}{r}
2x^3-3x^2-9x+5)\ 2x^4-x^3-10x^2-11x+8\ (x+1 \\
\underline{2x^4-3x^3-9x^2+5x} \\
2x^3-x^2-16x+8 \\
\underline{2x^3-3x^2-9x+5} \\
2x^2-7x+3=r.
\end{array}$$

$$\begin{array}{r}
r=2x^2-7x+3)\ 2x^3-3x^2-9x+5\ (x+2 \\
\underline{2x^3-7x^2+3x} \\
4x^2-12x+5 \\
\underline{4x^2-14x+6} \\
2x-1
\end{array}$$

$$\begin{array}{r}
2x-1)\ 2x^2-7x+3\ (x-3 \\
\underline{2x^2-x} \\
-6x+3 \\
\underline{-6x+3} \\
0\ \dagger
\end{array}$$

Here, by dividing the numerator by the denominator, we get for quotient $x+1$ and for remainder $2x^2-7x+3$. We then take this remainder as divisor, and the former divisor as dividend, and we get as quotient $x+2$, with $2x-1$ remaining. In the next place, taking $2x-1$ as divisor, and the last divisor as dividend, we find for quotient $x-3$ with no remainder. Hence, $2x-1$ is the common measure; and if we divide the terms of the given fraction by it, we get for answer $\frac{x^3-5x-8}{x^2-x-5}$.

---

* Should this factor be common to both the dividend and the divisor, they must both be divided by it.

† In this question, as well as in many others of a similar kind, the me-

The reason of this process will be seen by examining it in a reversed order. Thus, from the third operation in division we see that $2x^2-7x+3$, which for brevity we may call r, is measured by $2x-1$, containing it $x-3$ times without remainder. The second division shows that the given denominator D is equal to $r(x+2)+2x-1$, which (§ 77.) is divisible by $2x-1$, since, as we have already seen, r is divisible by it. From the first division, again, we see that the numerator N is equal to $D(x+1)+r$; and this (§ 77.) is evidently divisible by $2x-1$, since this quantity has been shown to be a measure both of D and r.

That no quantity of a higher order than $2x-1$ can measure N and D may be thus shown by means of § 77. Suppose, if possible, that a factor of the form, ax^2+bx+c, can measure them; then, since this factor measures D, it must measure $D(x+1)$; and measuring N, it must measure r, which is the difference of N and $D(x+1)$. Measuring, therefore, r, it must measure $r(x+2)$; and measuring D and $r(x+2)$, it must measure $2x-1$, which is

---

thod of detached coefficients may be employed with much advantage. By using also, alternately, the foreign and the British mode of arrangement, the *type* or form of the operation will be rendered more convenient. The work in this way will be as follows, without, however, all the contraction and condensation of which it is susceptible according to § 50., &c. In this way the operation is purely arithmetical, and the pupil is freed, till the conclusion, from all trouble in writing x and its powers, and in considering what those powers are to be. The divisor which leaves no remainder is $2-1$, or $2x-1$, the same that was found by the lengthened process. It is scarcely necessary to remark, that the dividing of the given numerator and denominator by $2x-1$ will also be most easily effected by using only the coefficients.

```
2 - 1 - 10 - 11    8 | 2 - 3 - 9    5
2 - 3 -  9    5      |_______________
_________________    | 1   1
    2 -  1 - 16    8 |
    2 -  3 -  9    5
    _______________
         2 -  7    3) 2 - 3 - 9    5 (1   2
                      2 - 7   3
                      _________
                          4 - 12   5
                          4 - 14   6
                          _________
              2 - 7       3 | 2 - 1
              2 - 1         |______
              _____         | 1 - 3
                  - 6     3
                  - 6     3
                  _________
                          0
```

their difference. Now, this is evidently absurd; as it is plain, that a factor of a higher degree cannot be contained in one of a lower.

Exam. 2. Find the greatest common measure of the quantities, $2x^3-15x+14$, and $x^4-15x^2+28x-12$; and thus reduce the fraction $\frac{x^4-15x^2+28x-12}{2x^3-15x+14}$ to its lowest terms.

$$
\begin{array}{rl}
2x^3-15x+14) & x^4-15x^2+28x-12 \\
 & \underline{2} \\
 & 2x^4-30x^2+56x-24\,(x \\
 & \underline{2x^4-15x^2+14x} \\
-3) & \underline{-15x^2+42x-24} \\
 & 5x^2-14x+8
\end{array}
$$

$$
\begin{array}{rl}
5x^2-14x+8) & 2x^3-15x+14 \\
 & \underline{5} \\
 & 10x^3-75x+70\,(2x \\
 & \underline{10x^3-28x^2+16x} \\
7) & \underline{28x^2-91x+70} \\
 & 4x^2-13x+10 \\
 & \underline{5} \\
 & 20x^2-65x+50\,(4 \\
 & \underline{20x^2-56x+32} \\
-9) & \underline{-9x+18} \\
 & x-2
\end{array}
$$

$$
\begin{array}{rl}
x-2) & 5x^2-14x+8\,(5x-4 \\
 & \underline{5x^2-10x} \\
 & -4x+8 \\
 & \underline{-4x+8} \\
 & 0
\end{array}
$$

In working this example, we multiply the dividend (§ 81.) by 2, so that we may have x, instead of $\frac{1}{2}x$ in the quotient. It is then easily seen, that the remainder is divisible by -3; we perform (§ 81.) this division, therefore, and take the quotient, $5x^2-14x+8$, as divisor, and the former divisor as dividend. To prevent the quotient, however, from being fractional, we multiply (§ 81.) this dividend by 5, and we thus get $2x$ for quotient, with the remainder, $28x^2-91x+70$. This, we readily see, can be divided by 7, and (§ 81.) we divide it accordingly, to keep the coefficients as small as possible. The result we multiply by 5, to prevent the next part of the quotient from being fractional; and we get 4 as quotient, with $-9x+18$ as remainder. This last being divided (§ 81.) by -9 gives $x-2$: and this, as it is contained without remainder, according to the annexed operation, in the last divisor, is the common measure required. Lastly, by dividing the terms of the given fraction by this, we get $x^3+2x^2-11x-6$ as numerator, and $2x^2+4x-7$ as denominator, of the answer. The reason of the various steps of the process is evident from the references given above.*

82. The greatest common measure of two quantities is often more easily found by the following rule†, than by the one already given. 1. Write the coefficients in two lines in succession, calling the one (*a*) and the other (*b*). 2. If necessary, multiply (*a*) and (*b*), so that the first terms of the results may be equal‡: then add or subtract so as to destroy those terms, and let the result, simplified, if possible, by having its terms divided by a common measure, be called (*c*). 3. Proceed in a similar manner with regard to the last terms of (*a*) and (*b*), and call the result (*d*). If (*c*) and (*d*) be the same, their terms are the coefficients

---

* The work, according to the method pointed out in the note to the last exercise, is given in the margin. The operation in this way is short and simple; and, after a little practice, the pupil will prefer it to the ordinary process employed in the text.

```
1   0-15  28-12 | 2   0-15  14
2               |-------------
--------------- | 1
2   0-30  56-24 |
2   0-15  14
 -----------
 - 3)-15  42-24
   -----------
        5-14   8)2 ·  0-15  14
                 5
               ---------------
               10   0-75  70(2
               10-28  16
               ---------
                 7) 28-91  70
                   ---------
                     4-13  10
                     5
                    -------
                   20-65  50(4
                   20-56  32
                  ----------
                 -9)- 9   18
                    ---------
             5-14     8 | 1 - 2
             5-10       |-------
            -----       | 5 - 4
              - 4   8
              - 4   8
             --------
                 0
```

† By working Examples 3, 4, and 5. by the common rule (§ 81.), and Examples 1. and 2. by the rule here given, which the author believes to be new, the student will find the latter to be superior, in general, in a very considerable degree, in point of brevity and facility. It has often in particular the advantage, especially in lengthened operations, of preventing so high numbers from arising in the process, as those which occur in the common method.

‡ The multipliers will be the coefficients of the two terms interchanged, or numbers got by dividing those coefficients by a common measure. Thus, in Exam. 3., to get results from (*a*) and (*b*) having their last terms equal, we might multiply (*a*) by 12 and (*b*) by 8. It is better, however, to divide these by 4, and to use the quotients 3 and 2. To get equal first terms, we might multiply (*a*) by 6 and (*b*) by 3; but, as the first of these is double of the other, it is simpler, and is sufficient, to double (*a*) and retain (*b*) unchanged.

The letters (*a*), (*b*), (*c*), &c., are used here merely for simplying the rule and the illustrations, and need not be employed in practice.

of the required measure. It sometimes but rarely happens, that the expression thus obtained will be found on trial not to be the common measure; when this is so, that quantity must be combined, according to the rule, with one of the original quantities, or with some quantity obtained from them. If (*c*) and (*d*) be not the same, treat them as was done with (*a*) and (*b*), and thus proceed till two are found which are the same.

In practice it is often better to find only (*c*), or only (*d*), getting the one that can be more easily found, and then to use it with (*a*) or (*b*). Like variations may be made in other parts of the work; and the general rule should be, to operate on lines containing the same, or most nearly the same, number of terms.

Exam. 3. Find the greatest common measure of the two quantities, $3x^3+2x^2-14x+8$, and $6x^3-11x^2+13x-12$.

In this operation, besides the explanations given in the margin, it may be remarked that (*c*) is found by taking (*b*) from the line below it, and (*d*) by adding the two lines preceding it. The line above (*e*) is got by subtracting the upper of the two lines next above it from the lower, and the last line of all is found by adding together the two lines next preceding it. The common measure required is $3x-4$, since the last two parts of the operations give 3 −4 equally.

$$\begin{array}{rrrrl}
3 & 2 & -14 & 8 & ..(a)\\
6 & -11 & 13 & -12 & ..(b)\\
6 & 4 & -28 & 16 & ..(a)\times 2\\ \hline
 & 15 & -41 & 28 & ..(c)\\ \hline
9 & 6 & -42 & 24 & ..(a)\times 3\\
12 & -22 & 26 & -24 & ..(b)\times 2\\ \hline
21 & -16 & -16 & & .....(d)\\ \hline
105 & -287 & 196 & & .....(c)\times 7\\
105 & -80 & -80 & & .....(d)\times 5\\ \hline
69) & 207 & -276 & & \\ \hline
 & 3 & -4 & & .....(e)\\ \hline
60 & -164 & 112 & & .....(c)\times 4\\
147 & -112 & -112 & & .....(d)\times 7\\ \hline
207 & -276, & \text{as before.} & &
\end{array}$$

The reason of the process will appear from considering that, according to §§ 77. and 78., every line, when the powers *x* are supplied, is divisible by the required measure. Thus, since (*a*) and (*b*) are divisible by it, the third line, which is the double of (*b*) is also (§ 77.) divisible by it; and so (by the same §) is (*c*), which is the difference of two quantities of which it is a measure. When the powers of *x* are supplied, (*d*) becomes $21x^3-16x^2-16x$; and (§ 78.) this may be divided by *x*, so as to become of the same order with (*c*). By each of the two concluding parts of the operation, we get $207x-276$; and the terms of this have the common simple factor 69, by which (§ 78.) we may divide them, as the given quantities have no such factor: and here the work evidently terminates.

Exam. 4. Find the greatest common factor of $3x^4+5x^3y+9x^2y^2+xy^3+6y^4$, and $2x^4+5x^3y+5x^2y^2-3xy^3-9y^4$.

In the work of this example, in addition to the explanations given in the margin. it may be stated, that to find (*c*), the upper of the two lines next above it is taken from the lower; that the same is done in the line before (*e*); and that at the conclusion (*c*) is taken from 13 times (*d*). Then, the coefficients being found in (*e*) to be 1, 2, and 3, it follows, that the common measure is $x^2+2xy+3y^2$.

$$\begin{array}{rrrrrrl}
 & 3 & 5 & 9 & 1 & 6 & ..(a) \\
 & 2 & 5 & 5 & -3 & -9 & ..(b) \\ \hline
 & 6 & 10 & 18 & 2 & 12 & ..(a)\times 2 \\
 & 6 & 15 & 15 & -9 & -27 & ..(b)\times 3 \\ \hline
 & & 5 & -3 & -11 & -39 & ..(c) \\ \hline
 & 9 & 15 & 27 & 3 & 18 & ..(a)\times 3 \\
 & 4 & 10 & 10 & -6 & -18 & ..(b)\times 2 \\ \hline
 & 13 & 25 & 37 & -3 & & ....(d) \\ \hline
 & 65 & -39 & -143 & -507 & & ....(c)\times 13 \\
 & 65 & 125 & 185 & -15 & & ....(d)\times 5 \\ \hline
164) & 164 & 328 & 492 & & & \\ \hline
 & 1 & 2 & 3 & & & ....(e) \\ \hline
 & 169 & 325 & 481 & -39 & & ..(d)\times 13 \\ \hline
 & 164 & 328 & 492, & \text{as before.} & &
\end{array}$$

Exam. 5. Reduce the fraction $\frac{x^4-8x^2+x-6}{x^3+6x^2+10x+3}$, to its lowest terms.

Here, the third line is got by taking (*a*) from (*b*), and (*d*) by taking (*b*) from (*c*). The line (*e*) and the last line but one are respectively the sums of the two lines preceding them. We might have found the line (*f*) over again by multiplying (*d*) by 9, and (*e*) by 2, and subtracting. In this case, however, as

$$\begin{array}{rrrrrrl}
 & 1 & 0 & -8 & 1 & -6 & ..(a) \\
 & 1 & 6 & 10 & 3 & & ......(b) \\ \hline
2) & 6 & 18 & 2 & 6 & & \\ \hline
 & 3 & 9 & 1 & 3 & & ..(c) \\ \hline
 & 2 & 3 & -9 & & & ......(d) \\ \hline
 & 9 & 27 & 3 & 9 & & ......(c)\times 3 \\ \hline
 & 9 & 29 & 6 & & & ....(e) \\ \hline
 & 4 & 6 & -18 & & & ....(d)\times 2 \\
 & 27 & 87 & 18 & & & ....(e)\times 3 \\ \hline
31) & 31 & 93 & & & & \\ \hline
 & 1 & 3 & & & & ....(f)
\end{array}$$

well as in similar ones, except for completing the operation, this last process is unnecessary; as it is plain, that, if there be a binomial factor, it must be what is thus obtained. In the present instance, we find that (f), that is, $x+3$, measures both terms of the given fraction, and reduces it to $\dfrac{x^3-3x^2+x-2}{x^2+3x+1}$.

83. *To find the greatest common measure of more than two quantities*, find the greatest common measure of two of them; the greatest common measure of the result and a third; and so on, if there be more.

Thus, if x^4-a^4, bx^3-a^2bx, and x^3+a^3 be proposed, we find that x^2-a^2 is the greatest common measure of the first and second, and that $x+a$ is the greatest common measure of x^2-a^2 and x^3+a^3. Hence $x+a$ is the greatest common measure of the three proposed quantities.

To illustrate this, since x^2-a^2 is the greatest common measure of the first and second, these two may be put under the forms, $(x^2+a^2)\,(x^2-a^2)$ and $bx(x^2-a^2)$, where x^2+a^2 and bx cannot have a common factor. Again, since $x+a$ is the greatest common measure of x^2-a^2 and x^3+a^3, the second and third may be put under the forms, $bx(x-a)(x+a)$ and $(x^2-ax+a^2)(x+a)$, and the first may be written $(x^2+a^2)(x-a)(x+a)$, where $x+a$ is obviously the only factor that is or can be common to all the three quantities.

Exercises. Find in each of the following fractions, the greatest common measure of its numerator and denominator, and reduce the fraction to its lowest terms.

14. $\dfrac{24a^2b^3x^5}{16abx^2y}$. *Ans.* $8abx^2$ and $\dfrac{3ab^2x^3}{2y}$.

15. $\dfrac{6x^5y^3}{36x^3y^5}$. *Ans.* $6x^3y^3$ and $\dfrac{x^2}{6y^2}$.

16. $\dfrac{4ab^2c^4}{6a^2b^2c^2}$. *Ans.* $2ab^2c^2$ and $\dfrac{2c^2}{3a}$.

17. $\dfrac{abcde}{bcdef}$. *Ans.* $bcde$ and $\dfrac{a}{f}$.

18. $\dfrac{x^3+a^3}{x^2+2ax+a^2}$. *Ans.* $x+a$ and $\dfrac{x^2-ax+a^2}{x+a}$.

19. $\dfrac{x^3-a^3}{ax-a^2}$. *Ans.* $x-a$ and $\dfrac{x^2+ax+a^2}{a}$.

20. $\dfrac{8a^3-27b^3}{4a^2-9b^2}$. *Ans.* $2a-3b$ and $\dfrac{4a^2+6ab+9b^2}{2a+3b}$.

21. $\dfrac{3x^5+2x^4-x^3-x^2+2x+3}{x^3-5x^2-5x+1}$.

Ans. $x+1$ and $\dfrac{3x^4-x^3-x+3}{x^2-6x+1}$.

22. $\dfrac{x^{2n+1}+a^{2n+1}}{x^{2m+1}+a^{2m+1}}$.

Ans. $x+a$ and $\dfrac{x^{2n}-ax^{2n-1}+a^2x^{2n-2}-\ \ldots\ -a^{2n-1}x+a^{2n}}{x^{2m}-ax^{2m-1}+a^2x^{2m-2}-\ \ldots\ -a^{2m-1}x+a^{2m}}$.

23. $\dfrac{x^4-x^2-2x+2}{2x^3-x-1}$. *Ans.* $x-1$ and $\dfrac{x^3+x^2-2}{2x^2+2x+1}$.

24. $\dfrac{18x^4-5x^2+44x-5}{3x^4+20x^3-57x^2+80x-50}$.

Ans. $3x^2-4x+5$ and $\dfrac{6x^2+8x-1}{x^2+8x-10}$.

25. $\dfrac{16x^5-17x^3-26x^2+18}{64x^5-27x^2}$.

Ans. $4x-3$ and $\dfrac{4x^4+3x^3-2x^2-8x-6}{16x^4+12x^3+9x^2}$.

26. $\dfrac{16x^4-53x^2+45x+6}{8x^4-30x^3+31x^2-12}$.

Ans. $4x^2-9x+6$ and $\dfrac{4x^2+9x+1}{2x^2-3x-2}$.

27. $\dfrac{24x^5-22x^4-14x^3+24x^2-8x}{18x^5-18x^4-14x^3+30x^2-12x}$.

Ans. $6x^2-4x$ and $\dfrac{4x^3-x^2-3x+2}{3x^3-x^2-3x+3}$.

28. $\dfrac{x^4-5x^3+4x^2+3x+9}{4x^3-15x^2+8x+3}$. *Ans.* $x-3$ and $\dfrac{x^3-2x^2-2x-3}{4x^2-3x-1}$.

29. $\dfrac{4x^4-12x^3+5x^2+14x-12}{6x^4-11x^3+9x^2-13x+6}$.

Ans. $2x-3$ and $\dfrac{2x^3-3x^2-2x+4}{3x^3-x^2+3x-2}$.

30. $\dfrac{9x^6+11x^4+18x^3+42x^2-8}{18x^6-5x^4-18x^3+33x^2-4}$.

Ans. $3x^3+4x^2+3x-2$ and $\dfrac{3x^3-4x^2+6x+}{6x^3-8x^2+3x+2}$.

84. *To add fractions together, which have a common denominator;* add their numerators together, and below the sum write the common denominator.*

Thus the sum of $\frac{x}{a}, \frac{y}{a}, \frac{z}{a}$, is $\frac{x+y+z}{a}$; and the sum $\frac{x}{x+a}$ and $\frac{a}{x+a}$ is $\frac{x+a}{x+a}$, or 1.

85. *To add fractions together which have different denominators;* reduce them (§ 73. or 74.) to equivalent ones having the same denominator, and add the results by the last rule.

Thus, the sum of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad+bc}{bd}$, and that of $\frac{a}{b}$ and $\frac{b}{a}$ is $\frac{a^2+b^2}{ab}$. In like manner, the sum of $\frac{1}{x-a}$ and $\frac{1}{x+a}$, or of their equivalents $\frac{x+a}{x^2-a^2}$ and $\frac{x-a}{x^2-a^2}$ is $\frac{2x}{x^2-a^2}$.

Exercises. Add together the following sets of fractions.

31. $\frac{1}{x}, \frac{2}{y}$, and $\frac{3}{z}$. *Ans.* $\frac{yz+2xz+3xy}{xyz}$.

32. $\frac{1}{x}, \frac{1}{2y}$, and $\frac{1}{3z}$. *Ans.* $\frac{6yz+3xz+2xy}{6xyz}$.

33. $\frac{x}{x+a}$ and $\frac{a}{x-a}$. *Ans.* $\frac{x^2+a^2}{x^2-a^2}$.

34. $\frac{1}{a^3b^2}$ and $\frac{3}{a^2b^3}$. *Ans.* $\frac{3a+b}{a^3b^3}$.

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* The reason of this rule is so obvious as scarcely to require illustration. It may be proved thus, however; let $F_1=\frac{x}{a}$, and $F_2=\frac{y}{a}$. Multiply each of these by a, and add the results; then $F_1a+F_2a=x+y$. Hence, by dividing by a, we get $F_1+F_2=\frac{x+y}{a}$. The rule for subtraction (§ 86.) would be proved in a similar manner.

35. $\frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \ldots, \frac{1}{a^{n-1}}, \frac{1}{a^n}$

Ans. $\frac{a^{n-1}+a^{n-2}+a^{n-3}+\ldots+a+1}{a^n}$.

36. $\frac{1}{x}, \frac{1}{x+1}, \frac{2}{x-3}$. *Ans.* $\frac{4x^2-3x-3}{x^3-2x^2-3x}$.

37. $2a-\frac{3x}{b}$ and $4a-\frac{x}{c}$. *Ans.* $6a-\frac{3cx+bx}{bc}$.

38. $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$. *Ans.* $\frac{2x^2+2y^2}{x^2-y^2}$.

39. $\frac{p_1}{x+a_1}, \frac{p_2}{x+a_2}$, and $\frac{p_3}{x+a_3}$. *Ans.* Numerator $=(p_1+p_2+p_3)x^2+\{p_1(a_2+a_3)+p_2(a_1+a_3)+p_3(a_1+a_2)\}x+p_1a_2a_3+p_2a_1a_3+p_3a_1a_2$, and denominator $=(x+a_1)(x+a_2)(x+a_3)$.

40. $\frac{1}{x+1}, \frac{2}{x+2}, \frac{3}{x+3}$, and $\frac{4}{x+4}$.

Ans. $\frac{10x^3+70x^2+150x+96}{x^4+10x^3+35x^2+50x+24}$.

41. $\frac{1}{x-1}, \frac{2}{2x-1}, \frac{3}{3x-1}$, and $\frac{4}{4x-1}$.

Ans. $\frac{96x^3-150x^2+70x-10}{24x^4-50x^3+35x^2-10x+1}$.*

86. *To subtract a fraction from another which has the same denominator;* from the numerator of the second, take the numerator of the first, and below the remainder write the common denominator.

Thus, if from $\frac{x}{a}$ we take $\frac{y}{a}$, the remainder is $\frac{x-y}{a}$; and if $\frac{a}{x-a}$ be taken from $\frac{x}{x-a}$, there remains $\frac{x-a}{x-a}$ or 1.

87. *To subtract a fraction from another which has a different denominator;* reduce them (§ 73. or 74.) to equivalent ones having the same denominator, and proceed with the results according to § 86.

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* Let the student compare this exercise and the last.

Thus, $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$; $\frac{a}{b}-\frac{b}{a}=\frac{a^2-b^2}{ab}$; and $\frac{1}{x-a}-\frac{1}{x+a}$, or, what is equivalent, $\frac{x+a}{x^2-a^2}-\frac{x-a}{x^2-a^2}=\frac{2a}{x^2-a^2}$.

Exercises. In each of the following exercises, take the second quantity from the first.

42. $\frac{1}{x}$ and $\frac{2}{y}$. *Ans.* $\frac{y-2x}{xy}$.

43. $\frac{1}{x}$ and $\frac{1}{2y}$. *Ans.* $\frac{2y-x}{2xy}$.

44. $\frac{a^2}{xy}$ and $\frac{b^2}{yz}$. *Ans.* $\frac{a^2z-b^2x}{xyz}$.

45. $\frac{1}{a^3b^2}$ and $\frac{3}{a^2b^3}$. *Ans.* $\frac{b-3a}{a^3b^3}$

46. $\frac{1}{x}$ and $\frac{1}{x+1}$. *Ans.* $\frac{1}{x^2+x}$.

47. $7x-\frac{a^2}{b}$ and $3x-\frac{a^3}{b^2}$. *Ans.* $4x-\frac{a^2b-a^3}{b^2}$.

48. $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$. *Ans.* $\frac{4xy}{x^2-y^2}$.

49. 1 and $\frac{x^2-y^2}{x^2+y^2}$. *Ans.* $\frac{2y^2}{x^2+y^2}$.

88. *To find the product of two or more fractions*; find the product of their numerators for the numerator of the answer, and the product of their denominators for its denominator.

Thus, the product of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$, is $\frac{ace}{bdf}$.

To prove this, let $F_1=\frac{a}{b}$, $F_2=\frac{c}{d}$, and $F_3=\frac{e}{f}$. Then, by multiplying these severally by b, d, and f, we get $F_1b=a$, $F_2d=c$, and $F_3f=e$; whence, by multiplication, we get $F_1F_2F_3bdf=ace$; and dividing by bdf, we get $F_1F_2F_3$, the product of the three fractions, equal to $\frac{ace}{bdf}$: and a similar proof may be given in every case.

89. In multiplying fractions, it is better, on many occasions, merely to *indicate* the operation at first, and then to simplify the

result by dividing the numerator and denominator by any factor that may belong to them in common. After this, any operations that are indicated may be performed, if it seem advantageous.

Thus, $\dfrac{x^2-a^2}{xy} \times \dfrac{ax}{xy+ay} = \dfrac{(x+a)(x-a)ax}{xyy(x+a)} = \dfrac{(x-a)a}{yy} = \dfrac{ax-a^2}{y^2}$.

It may be remarked also, that, in both multiplication and division, where there are mixed quantities, they should in general be reduced to fractions by § 75.

Exercises. Find the products of the fractions given in each of the following exercises.

50. $\dfrac{a^2}{b}$, $\dfrac{c^3}{d^2}$, and $\dfrac{e^4}{f^3}$. *Ans.* $\dfrac{a^2c^3e^4}{bd^2f^3}$.

51. $\dfrac{a^2}{b}$, $\dfrac{b^3}{c^2}$, and $\dfrac{c^4}{d^3}$. *Ans.* $\dfrac{a^2b^2c^2}{d^3}$.

52. $\dfrac{x^2-ax+a^2}{x^2+ax+a^2}$ and $\dfrac{x+a}{x-a}$. *Ans.* $\dfrac{x^3+a^3}{x^3-a^3}$.

53. $\dfrac{3x^2-4x+1}{x^2+4x-3}$ and $\dfrac{3x+4}{x-4}$. *Ans.* $\dfrac{9x^3-13x+4}{x^3-19x+12}$.

54. $\dfrac{x^2-4a^2}{x-a}$ and $\dfrac{x^3-a^3}{x+2a}$. *Ans.* $x^3-ax^2-a^2x+2a^3$.

55. $\dfrac{a^4}{bx-ab}$ and $\dfrac{b^2}{x-a}$. *Ans.* $\dfrac{a^4b}{(x-a)^2}$.

56. $\dfrac{1+a+a^2}{1-b+b^2}$ and $\dfrac{1-a}{1+b}$. *Ans.* $\dfrac{1-a^3}{1+b^3}$.

57. $1-\dfrac{x-y}{x+y}$ and $2+\dfrac{2y}{x-y}$. *Ans.* $\dfrac{4xy}{x^2-y^2}$.

58. $\dfrac{6x^3-4x^2-9x+6}{6x^3+4x^2-9x+6}$ and $\dfrac{3x^3+2x^2+9x+6}{6x^3-4x^2+3x+2}$.

Ans. $\dfrac{18x^6+19x^4-93x^2+36}{36x^6-52x^4+96x^3-43x^2+12}$.

90. *To divide one fraction by another; invert* the divisor, that is, take its numerator as denominator, and its denominator as numerator, and proceed as in multiplying fractions.*

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* More generally, any quantity, whole or fractional, is divided by another, by multiplying the former by the reciprocal of the latter. (See

Thus, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

To prove this, let $F_1 = \frac{a}{b}$, and $F_2 = \frac{c}{d}$. Then, $F_1 b = a$, and $F_2 d = c$. Multiply the first of these by d, and the second by b; then $F_1 bd = ad$, and $F_2 bd = bc$. Hence, by division,

$$\frac{F_1 bd}{F_2 bd}, \text{ or § 68. } \frac{F_1}{F_2} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}.$$

and, therefore, the foregoing rule is correct, as it gives the same result that we have thus found.

91. The principle pointed out in § 89. is employed with as much advantage in division, as in multiplication, of fractions.

Thus, $\frac{a^2+ax}{b^2-x^2} \div \frac{a^3-ax^2}{b-x} = \frac{a(a+x)}{(b+x)(b-x)} \times \frac{b-x}{a(a+x)(a-x)}$
$= \frac{1}{(b+x)(a-x)}$.

92. *A complex fraction*, that is, a fraction having its numerator or denominator, or both, fractional, *is reduced to a simple one*, by dividing its numerator by its denominator, in the manner that has been pointed out.

Thus, if the fraction $\frac{x+\frac{3}{4}}{2x-\frac{5}{6}}$ be proposed, the numerator and denominator become respectively (by § 75.) $\frac{4x+3}{4}$ and $\frac{12x-5}{6}$; and, by dividing the first of these by the second, we get $\frac{12x+9}{24x-10}$, the simple fraction required.

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the note to § 46.) Thus, if we divide a by b, the quotient is $\frac{a}{b}$, or, as it may be written, $a \, . \, \frac{1}{b}$.

This principle is sometimes useful in arithmetical computations, especially when the same number is to be often used as a divisor. Thus, as a useful instance, since an imperial gallon consists of 277·274 cubic inches, it follows, that any number of cubic inches will be reduced to gallons by dividing by that number. Instead of doing this, however, we may multiply the number of inches by ·00360654, which is found by dividing 1 by 277·274. In this instance, even by multiplying by ·0036, that is, by multiplying by 36, and cutting off four figures as a decimal, we should get the true number of gallons very nearly; and if much accuracy were required, we might add to the result $\frac{1}{500}$ (more strictly $\frac{1}{550}$) of itself.

The same result would be obtained by multiplying the numerator and denominator of the given complex fraction by 12, the least common multiple of the denominators 4 and 6; and the same may be done in other cases.*

Exercises. In each of the following exercises, divide the first of the two quantities by the second.

59. $\frac{ab^2c^3}{x^2y}$ and $\frac{a^3b^2c}{xy^2}$. *Ans.* $\frac{c^2y}{a^2x}$.

60. $\frac{2x-1}{x+2}$ and $\frac{x-3}{3x+1}$. *Ans.* $\frac{6x^2-x-1}{x^2-x-6}$.

61. $\frac{y}{y-2}$ and $\frac{y-1}{y^2+2}$. *Ans.* $\frac{y^3+2y}{y^2-3y+2}$.

62. $\frac{a+b}{a+c}$ and $\frac{a-c}{a-b}$. *Ans.* $\frac{a^2-b^2}{a^2-c^2}$.

63. $\frac{x^3-a^2x}{a^2}$ and $\frac{ax-a^2}{x}$. *Ans.* $\frac{x^3+ax^2}{a^3}$.

64. $\frac{x^m}{y^n}$ and $\frac{x^n}{y^m}$. *Ans.* $(xy)^{m-n}$.

65. $1+\frac{1}{a}$ and $1-\frac{1}{a^2}$. *Ans.* $\frac{a}{a-1}$.

66. $\frac{x+b}{x}$ and $\frac{x^2-b^2}{x^2y}$ *Ans.* $\frac{xy}{x-b}$.

67. $\frac{ax^2+ab^2}{x^2-ax}$ and $\frac{a^2x+a^3}{x^3-b^2x}$. *Ans.* $\frac{x^4-b^4}{ax^2-a^3}$.

68. $1-\frac{x^2}{x^2+a^2}$ and $1+\frac{a^2}{x^2-a^2}$. *Ans.* $\frac{a^2x^2-a^4}{x^4+a^2x^2}$.

69. $x^2+4a^2+\frac{15a^4}{x^2-4a^2}$ and $x+2a+\frac{3a^2}{x-2a}$. *Ans.* $\frac{x^2+a^2}{x+2a}$.

70. $9x^2-28+\frac{4}{x^2}$ and $3x-4-\frac{2}{x}$. *Ans.* $\frac{3x^2+4x-2}{x}$.

93. In some instances, operations on fractional quantities are facilitated by the use of negative indices. This is particularly

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* This latter method is virtually the same as reducing the numerator and denominator to equivalent fractions having a common denominator; and then rejecting that denominator, and dividing the one numerator by the other; a method which may be employed in many instances with advantage.

the case, when the denominators are monomials. Thus, if it be required to multiply $\frac{x}{y}-\frac{2x^2}{y^2}-\frac{3x^3}{y^3}$ by $\frac{x}{y^2}+\frac{2x^2}{y^3}-\frac{3x^3}{y^4}$, these quantities and the whole work may stand as in the margin, and the process will be seen to be simple and easy. If, however, we use only the coefficients, as in the second process, the work is still much easier. This, indeed, is always the case, when the quantities without their coefficients proceed by indices having a common difference.

$$\begin{array}{lllll}
xy^{-1}-2x^2y^{-2}-3x^3y^{-3} & & & & \\
xy^{-2}+2x^2y^{-3}-3x^3y^{-4} & & & & \\
\hline
x^2y^{-3} & -2x^3y^{-4} & -3x^4y^{-5} & & \\
 & 2x^3y^{-4} & -4x^4y^{-5} & -6x^5y^{-6} & \\
 & & -3x^4y^{-5} & +6x^5y^{-6} & +9x^6y^{-7} \\
\hline
x^2y^{-3} & & -10x^4y^{-5} & & +9x^6y^{-7},
\end{array}$$

$$\text{or } \frac{x^2}{y^3}-\frac{10x^4}{y^5}+\frac{9x^6}{y^7}$$

$$\begin{array}{rrrrr}
1 & -2 & -3 & & \\
1 & 2 & -3 & & \\
\hline
1 & -2 & -3 & & \\
 & 2 & -4 & -6 & \\
 & & -3 & 6 & 9 \\
\hline
1 & 0 & -10 & 0 & 9
\end{array}$$

Miscellaneous Exercises regarding Fractions.

1. To the sum of $\frac{a}{b}$ and $\frac{b}{a}$, annex ± 2, and show that if the result be multiplied by ab, the product is $(a\pm b)^2$.

2. Multiply $\frac{a^3-a^2x}{a^3+x^3}$ by $\frac{2a+2x}{a-x}$. *Ans.* $\frac{2a^2}{a^2-ax+x^2}$.

3. Prove that $\frac{x}{x^2+ax+a^2}+\frac{2ax+a^2}{x^3-a^3}=\frac{1}{x-a}$.

4. Divide $\frac{x^3+y^3}{x^5-xy^4}$ by $\frac{x^2+2xy+y^2}{x^3+xy^2}$, exhibiting the quotient in its simplest form. *Ans.* $\frac{x^2-xy+y^2}{(x+y)(x^2-y^2)}$.

5. Prove that half the sum of the squares of $\frac{x+a}{x-a}$ and $\frac{x-a}{x+a}$ is equal to $\frac{8a^2x^2}{(x^2-a^2)^2}+1$.

6. Of the three fractions, $\frac{1}{x-a}, \frac{1}{x}$, and $\frac{1}{x+a}$, whether is the sum of the first and last greater or less than twice the second.

Ans. The sum of the extremes is greater by $\frac{2a^2}{x(x^2-a^2)}$, x being considered positive.

7. From the sum of the extremes of the four fractions,

$$\frac{1}{x-3a}, \frac{1}{x-a}, \frac{1}{x+a}, \text{ and } \frac{1}{x+3a},$$

subtract the sum of the means. *Ans.* $\frac{16a^2x}{(x^2-a^2)(x^2-9a^2)}$.*

8. From the sum of the squares of $\frac{1}{a-b}$ and $\frac{1}{a+b}$, take the square of $\frac{2b}{a^2-b^2}$. *Ans.* $\frac{2}{a^2-b^2}$.

9. Subtract the first of the following fractions from the second, the second from the third, &c.: $\frac{1}{x+1}, \frac{2}{2x+1}, \frac{3}{3x+1}, \frac{4}{4x+1}$, &c.

Ans. $\frac{1}{(x+1)(2x+1)}, \frac{1}{(2x+1)(3x+1)}, \frac{1}{(3x+1)(4x+1)}$, ..., $\frac{1}{\{(n-1)x+1\}\{nx+1\}}$.

10. From the first of the fractions in the answer to the last question take the second; from the second take the third; and so on.

Ans. $\frac{2x}{(x+1)(2x+1)(3x+1)}, \frac{2x}{(2x+1)(3x+1)(4x+1)}, \ldots, \frac{2x}{\{(n-2)x+1\}\{(n-1)x+1\}\{nx+1\}}$.†

11. From the first of the fractions in Exer. 6., take the second, from the second take the third, and take the second of the remainders from the first.

Ans. $\frac{a}{(x-a)x}, \frac{a}{x(x+a)}$, and $\frac{2a^2}{(x-a)x(x+a)}$.

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* This will be positive as long as x is positive, and greater than $3a$; but, if x be between a and $3a$, it will be negative. This will be seen from examining the denominator.

† The student may have further practice in subtraction by finding the successive differences of the quantities contained in this answer, the successive differences of the results, &c.

12. Divide the sum of $\frac{a+1}{a-1}$ and $\frac{a-1}{a+1}$ by their difference.

Ans. $\frac{a^2+1}{2a}=\frac{1}{2}(a+a^{-1})$.

13. In the last Exercise, change a into $\frac{a}{b}$, and thus prove, that, if the sum of $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ be divided by their difference, the quotient is $\frac{a^2+b^2}{2ab}$.

14. Divide $\frac{1}{1+x}+\frac{x}{1-x}$ by $\frac{1}{1-x}-\frac{x}{1+x}$. *Ans.* 1.

15. Divide $\frac{1}{1-x}+\frac{1}{1+x}$ by $\frac{1}{1-x}-\frac{1}{1+x}$. *Ans.* $\frac{1}{x}$.

16. Divide $\frac{1}{(x-2)(x-1)x(x+1)}$ by $\frac{1}{(x-1)x(x+1)(x+2)}$.

Ans. $\frac{x+2}{x-2}$.

17. Add together the two quantities given in the last exercise.

Ans. $\frac{2}{(x-2)(x-1)(x+1)(x+2)}$.

18. Subtract the latter of the same quantities from the former.

Ans. $\frac{4}{(x-2)(x-1)x(x+1)(x+2)}$.

CHAPTER V.

RADICALS, OR SURDS.*

94. Any quantity in which the extraction of a root that cannot be exactly determined is indicated, is called a *radical* or *surd*. Thus, $\sqrt{2}$, $\sqrt{(a+b)}$, $\sqrt[3]{(x^3+a^3)}$, &c., are radicals. Such quanti-

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* These quantities are generally called *surds* by English writers; while the French more appropriately term them *radicals*, from the Latin word

ties are sometimes called *irrational* or *incommensurable*; while, in reference to them, quantities in which no root is indicated, or none, at least, which cannot be assigned in a finite number of terms, are said to be *rational* or *commensurable*.* Such are a, b, $x-y$, $\sqrt{x^2}$, &c.

95. According to what we saw in multiplication, the third power of a^n is a^{n+n+n}, or a^{3n}; and, conversely, the third root of a^{3n} is a^n. To raise a^n, therefore, to the third power, we multiply n the index of the given power, by 3 the index of the required one; while, to extract the third root of a^{3n}, we divide the index $3n$ by 3. In like manner, the mth power of a^n is $a^{n+n+n+\cdots}$, or a^{mn}, n being obviously repeated m times. To find, therefore, the mth power of a^n, we multiply n by m; and, conversely, to extract the mth root of a^{mn}, we divide the index mn by m.† Hence, if instead of mn, we write p, we shall have the mth root of a^p equal to $a^{\frac{p}{m}}$; and in every case in which p is divisible by m, we get the root accurately in the primary sense of powers and roots. Thus, the fifth root of a^{15} is a^3, and the third root of a^{21}

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radix, a root, because of their expressing the roots of quantities; and the Germans distinguish them by a synonymous term, *wurzelgrössen* (*root quantities*). The introduction of such quantities has been almost entirely avoided in the preceding part of the work. In this section, their principal properties, and the modes in which operations are performed upon them will be investigated.

* These terms have their origin in the circumstance, that, between unity and quantities of the one kind, there are *common measures* or *ratios*, which can be exactly expressed in ordinary numbers; while for the others there are no such measures or ratios. Thus, $\frac{1}{4}$ is a common measure of 1 and $\frac{9}{4}$, and their ratio is that of 4 to 9; but, since the square root of 2 is 1·4142, &c., $\frac{1}{10}$ is a measure of 1 and 1·4, and the ratio of these is that of 10 to 14; $\frac{1}{100}$ is a measure of 1 and 1·41, and the ratio is that of 100 to 141; and thus we might proceed, by successive approximations, as far as we please: but we could find no measure of unity so small as to be contained exactly in the square root of 2, and therefore they have no ratio which can be expressed exactly in numbers. It is evident also, that all whole numbers are commensurable with one another, as unity is a common measure of them all. So also are all common fractions, both with respect to whole numbers and to one another. Thus, $\frac{1}{8}$ is a measure of $\frac{7}{8}$ and 2; and these numbers are in the ratio 7 to 16. In like manner, $\frac{5}{8}$ and $\frac{7}{12}$ have $\frac{1}{24}$ as a common measure, and are in the ratio of 15 to 14. Some of the statements in this note will be illustrated by the section on continued fractions.

† It will appear from the note to § 47. that this is applicable when n is a negative whole number, as well as when it is a positive one.

is a^{7}[*]. Even when, however, an index p is not divisible by m, and when, therefore, there is not an exact root, the same notation is adopted, the extraction of a root being indicated by a similar division of the index of the proposed quantity, so that *the mth root of a^p is expressed by $a^{\frac{p}{m}}$.* On this principle, the second, third, and fourth roots of a, or a^1, are expressed by $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, and $a^{\frac{1}{4}}$; and the third, fifth, and seventh roots of a^4, by $a^{\frac{4}{3}}$, $a^{\frac{4}{5}}$, and $a^{\frac{4}{7}}$. Hence also, it follows, conversely, that if $R=a^{\frac{m}{n}}$, we shall have, by raising both members to the nth power, $R^n=a^m$.[†]

96. Let $a^{\frac{1}{n}}=R$, and consequently, by § 95., $a=R^n$. By raising both members of each of these equations to the mth power, we get $(a^{\frac{1}{n}})^m=R^m$, and $a^m=R^{mn}$. The nth root of the latter of these is $a^{\frac{m}{n}}=R^m$, by § 95. We have thus found two expressions, each of which is equal to R^m; and by putting them equal to one another, we get $a^{\frac{m}{n}}=(a^{\frac{1}{n}})^m$: whence it appears, that $a^{\frac{m}{n}}$ may be regarded either as the nth root of the mth power of a, as it was defined in § 95., or as the mth power of the nth root of a. We thus see, that, in the primary meaning (§ 10.) of the term *power*, $a^{\frac{m}{n}}$ is a power, not of a, but of $a^{\frac{1}{n}}$. It is attended with much advantage, however, to extend the meaning of the term *power*, so as still to call $a^{\frac{m}{n}}$ a power of the original quantity a: and we have thus a farther extension of the meaning of that term, in addition to the one given in § 46., regarding quantities having negative indices. It may be remarked, that $a^{\frac{m}{n}}$ may be regarded as a general expression, which will comprehend all powers, giving powers of the primitive kind, when m and n have like signs, and n is unity or a submultiple of m; and powers with negative indices, when m and n have opposite signs.

97. Let $R=a^{\frac{p}{m}}$: then $R^m=a^p$. Raise both members to the nth power: then $R^{mn}=a^{pn}$; whence, by extracting the mnth root, we get $R=a^{\frac{pn}{mn}}$; and, therefore, $a^{\frac{p}{m}}=a^{\frac{pn}{mn}}$. We thus see,

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[*] We shall find afterwards, that there are other roots of these and similar quantities, besides those given by the principle here explained. The ones here pointed out, however, are sufficient for our present purpose.

[†] The clumsy notation, $\sqrt[n]{a^m}$, still used by some writers to denote the nth root of the mth power of a ought to be laid aside.

that *the value of a power is not changed by multiplying or dividing the numerator and denominator of its index by the same quantity.*

Thus, $a^{\frac{2}{3}} = a^{\frac{4}{6}} = a^{\frac{10}{15}}$, and a^3 or $a^{\frac{3}{1}} = a^{\frac{6}{2}} = a^{\frac{15}{5}}$.

98. If n be a whole number, we have, by multiplication, $(ab)^n = ab.ab.ab.... = aaa.... \times bbb.... = a^n b^n$; so that *the nth power of ab is equal to the product of the nth powers of its factors;* and, conversely, *the nth root of $a^n b^n$ is ab, the product of the nth roots of its factors, a^n and b^n.*

99. Again, let $R = (ab)^{\frac{m}{n}}$ and consequently (§ 95.) $R^n = (ab)^m$, or, by what we have just seen, $R^n = a^m b^m$. Hence, by taking the nth root, according to § 98., we get $R = a^{\frac{m}{n}} b^{\frac{m}{n}}$: and therefore $(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} b^{\frac{m}{n}}$; which shows, that the proposition established in § 98. is true, when the index is fractional, as well as when it is an integer.

100. The principle established in §§ 98. and 99. is often of much use in simplifying and otherwise modifying radicals. Thus, since $288 = 144 \times 2$, we shall have $\sqrt{288} = \sqrt{144} \times \sqrt{2} = 12\sqrt{2}$. In like manner, since $112 = 16 \times 7$, we have $\sqrt{112} = \sqrt{16} \times \sqrt{7} = 4\sqrt{7}$; and since $320 = 64 \times 5$, it follows, that $\sqrt[3]{320} = \sqrt[3]{64} \times \sqrt[3]{5} = 4\sqrt[3]{5}$. It would be shown, in the same way, that $\sqrt[3]{10000} = 10\sqrt[3]{10}$.*

101. In a similar manner, many radical expressions may be simplified. Thus, $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$; $\sqrt[3]{a^3b^2} = \sqrt[3]{a^3} \times \sqrt[3]{b^2} = a\sqrt[3]{b^2}$; and $\sqrt{(a^2x - a^3)} = \sqrt{a^2} \times \sqrt{(x-a)} = a\sqrt{(x-a)}$. So, also, $\sqrt{(a^2b + 2ab^2 + b^3)} = \sqrt{(a^2 + 2ab + b^2)} \times \sqrt{b} = (a+b)\sqrt{b}$;

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* In this way, we may often be saved much trouble and labour in the extraction of roots by the use of a table of roots of moderate extent. This will be exemplified by means of the subjoined short table, which exhibits the square and cube roots of the several whole numbers, commencing with 2 and ending with 51. In Hutton's and various other works, such tables will be found, carried out to a much greater extent.

To exemplify the use of the table in relation to the examples in the text, we get from it $\sqrt{2} = 1{\cdot}4142136$; and multiplying this by 12, we obtain 16·9705632, the square root of 288. As, in the table, the roots are given true to the nearest figure in the last place of decimals, the foregoing result may err in the last figure by nearly 6, half the multiplier; and the same is always the case. In the present instance, the true root is 16·9705627. To work the next example, we have, by the table, $\sqrt{7} = 2{\cdot}6457513$; the product of which by 4 is 10·5830052, the square root of 112. In the third place, we get from the table the cube root of 5 equal to 1·709976; and multiplying this by 4, we find the cube root of and $$\sqrt{\frac{2a^3-3a^2b}{x^2-2ax+a^2}} = \sqrt{\frac{a^2}{x^2-2ax+a^2}} \times \sqrt{(2a-3b)} = \frac{a}{x-a}\sqrt{(2a-3b)}.$$

Exercises.

Extract the following roots, by means of the table in the note in this page.

	Answers.		*Answers.*
1. $\sqrt{75}$.	8·660254.	4. $\sqrt{1500}$.	38·729833.
2. $\sqrt{117}$.	10·8166539.	5. $\sqrt[3]{108}$.	4·762203.
3. $\sqrt{1728}$.	41·5692192.	6. $\sqrt[3]{686}$.	8·819447.

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320 to be 6·839904; and lastly, the cube root of 10 being 2·154435 by the table, we multiply this by 10, and find the cube root of 10000 to be 21·54435.

TABLE OF SQUARE AND CUBE ROOTS.

Num.	Square Root.	Cube Root.	Num.	Square Root.	Cube Root.
2	1·4142136	1·259921	27	5·1961524	3·000000
3	1·7320508	1·442250	28	5·2915026	3·036589
4	2·0000000	1·587401	29	5·3851648	3·072317
5	2·2360680	1·709976	30	5·4772256	3·107232
6	2·4494897	1·817121	31	5·5677644	3·141381
7	2·6457513	1·912933	32	5·6568542	3·174802
8	2·8284271	2·000000	33	5·7445626	3·207534
9	3·0000000	2·080084	34	5·8309519	3·239612
10	3·1622777	2·154435	35	5·9160798	3·271066
11	3·3166248	2·223980	36	6·0000000	3·301927
12	3·4641016	2·289428	37	6·0827625	3·332222
13	3·6055513	2·351335	38	6·1644140	3·361975
14	3·7416574	2·410142	39	6·2449980	3·391211
15	3·8729833	2·466212	40	6·3245553	3·419952
16	4·0000000	2·519842	41	6·4031242	3·448217
17	4·1231056	2·571282	42	6·4807407	3·476027
18	4·2426407	2·620741	43	6·5574385	3·503398
19	4·3588989	2·668402	44	6·6332496	3·530348
20	4·4721360	2·714418	45	6·7082039	3·556893
21	4·5825757	2·758923	46	6·7823300	3·583048
22	4·6904158	2·802039	47	6·8556546	3·608826
23	4·7958315	2·843867	48	6·9282032	3·634241
24	4·8989795	2·884499	49	7·0000000	3·659306
25	5·0000000	2·924018	50	7·0710678	3·684031
26	5·0990195	2·962496	51	7·1414284	3·708430

Exhibit the following expressions in their simplest forms.

7. $\sqrt{\frac{ab^2c^3}{d^4e^5}}$. Ans. $\frac{bc}{d^2e^2}\sqrt{\frac{ac}{e}}$, or $\frac{bc}{d^2e^3}\sqrt{ace}$.

8. $\sqrt[3]{\frac{a^4b-a^3b^2}{c^5}}$. Ans. $\frac{a}{c}\sqrt[3]{\frac{ab-b^2}{c^2}}$, or $\frac{a}{c^2}\sqrt[3]{bc(a-b)}$.

9. $\sqrt{(3a^2b+6ab^2+3b^3)}$. Ans. $(a+b)\sqrt{3b}$.

10. $\sqrt[3]{\frac{a^4-a^3}{b^4+b^5}}$.

Ans. $\frac{a}{b}\sqrt[3]{\frac{a-1}{b+b^2}}$, or $\frac{a}{b^2(b+1)}\sqrt[3]{(a-1)b^2(b+1)^2}$.

11. $\sqrt{\frac{ax^2-2ax+a}{x^3+2x^2+x}}$. Ans. $\frac{x-1}{x+1}\sqrt{\frac{a}{x}}$, or $\frac{x-1}{x(x+1)}\sqrt{ax}$.

12. $\sqrt{\frac{a^2b^2c^2-a^2c^4}{a^2b-2ab^2+b^3}}$.

Ans. $\frac{ac}{a-b}\sqrt{\frac{(b+c)(b-c)}{b}}$, or $\frac{ac}{b(a-b)}\sqrt{b(b+c)(b-c)}$.

102. A multiplier of a radical may be brought under the influence of its index by reversing the process of the last §. Thus, $2\sqrt{3}=\sqrt{4}\times\sqrt{3}=\sqrt{12}$: $3\sqrt[3]{2}=\sqrt[3]{27}\times\sqrt[3]{2}=\sqrt[3]{54}$; and $ab^{\frac{1}{n}}=(a^nb)^{\frac{1}{n}}$.*

103. *To add or subtract radical quantities,* when the radical part is the same in all; perform the addition or subtraction on their multipliers or coefficients, according to the rules for the addition or subtraction of other quantities, and prefix the result to the common radical part. If the radical parts be not the same, reduce the given quantities, by § 102., to their simplest forms, if they admit of such reduction. Then, if the radical parts be the same, proceed according to the rule just given: but if they be different, the quantities must be connected by the sign + or −, as the case may require.

Thus, the sum of $\sqrt{5}$, $2\sqrt{5}$, and $5\sqrt{5}$, is $8\sqrt{5}$; and the difference of $6\sqrt{3}$ and $\sqrt{3}$ is $5\sqrt{3}$. Also, the sum of $\sqrt[3]{192}$ and $\sqrt[3]{375}$ is $9\sqrt[3]{3}$; for, by § 102., the first of these is equivalent to $4\sqrt[3]{3}$, and the second to $5\sqrt[3]{3}$. The sum, however, of $\sqrt{2}$ and $\sqrt[3]{3}$ cannot be expressed otherwise than by connecting them by the sign $+$, thus $\sqrt{2}+\sqrt[3]{3}$; and, in like manner, the difference of $6\sqrt{3}$ and $5\sqrt{2}$ is $6\sqrt{3}-5\sqrt{2}$; the radicals not being capable of incorporation in either case. The following examples will illustrate this subject: —

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* This principle enables us to solve questions such as the following, without the actual extraction of the roots: Whether is $5\sqrt{2}$ or $4\sqrt{3}$ greater? By the principle above pointed out, $5\sqrt{2}$ is the same as $\sqrt{50}$, and $4\sqrt{3}$ the same as $\sqrt{48}$; the first of which is greater than the second, as 50 is greater than 48. In the same manner, it would be shown, that $9\sqrt{3}$ is greater than $11\sqrt{2}$; that $58\sqrt{2}$ is greater than $31\sqrt{7}$; and that $299\sqrt{151}$ is greater, by a very little, than $301\sqrt{149}$.

Exam. 1.

$$\begin{aligned} \sqrt{98}&=7\sqrt{2}\\ \sqrt{32}&=4\sqrt{2}\\ \sqrt{72}&=6\sqrt{2}\\ \hline \text{Sum}&=17\sqrt{2} \end{aligned}$$

Exam. 2.

$$\begin{aligned} \sqrt{(ax^2-2a^2x+a^3)}&=(x-a)\sqrt{a}\\ \sqrt{a^3}&=a\sqrt{a}\\ \hline \text{Sum}&=\quad x\sqrt{a} \end{aligned}$$

Exam. 3.

$$\begin{aligned} \sqrt{(x^3-ax^2)}&=x\sqrt{(x-a)}\\ \sqrt{(a^2x-a^3)}&=a\sqrt{(x-a)}\\ \hline \text{Diff.}&=(x-a)\sqrt{(x-a)}\\ &=(x-a)^{\frac{3}{2}} \end{aligned}$$

Exercises.

13. Add together $\sqrt{112}$, $\sqrt{175}$, and $\sqrt{343}$. *Ans.* $16\sqrt{7}$.

14. From $3\sqrt{44}$ take $\sqrt{275}$. *Ans.* $\sqrt{11}$.

15. Find the sum and difference of the cube roots of 108 and 32. *Ans.* $5\sqrt[3]{4}$ and $\sqrt[3]{4}$.

16. Add together the square roots of a^2x, b^2x, and c^2x.
Ans. $(a+b+c)\sqrt{x}$.

17. Find the sum and difference of $\sqrt{(x^3+2x^2y+xy^2)}$ and $\sqrt{(x^3-2x^2y+xy^2)}$. *Ans.* $2x\sqrt{x}$, or $2x^{\frac{3}{2}}$, and $2y\sqrt{x}$.

104. By means of the principles established in §§ 96 and 97, we may reduce radicals having different indices to equivalent ones having the same index. Thus, if we have $\sqrt{2}$ and $\sqrt[3]{3}$, or, as they may be better written in the present case, $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$, these quantities, by the principles referred to, are equivalent to $2^{\frac{3}{6}}$ and $3^{\frac{2}{6}}$, or $8^{\frac{1}{6}}$ and $9^{\frac{1}{6}}$, which have the same index. In like manner, $a^{\frac{m}{n}}$ and $b^{\frac{m'}{n'}}$ are equivalent to $a^{\frac{mn'}{nn'}}$ and $b^{\frac{m'n}{nn'}}$, or to $(a^{mn'})^{\frac{1}{nn'}}$ and $(b^{m'n})^{\frac{1}{nn'}}$. *To reduce*, therefore, *radicals having different indices to equivalent ones having the same index*, reduce the given indices (§ 73. or 74.) to equivalent ones having the same denominator, and

raise the quantities affected by the original indices to the powers denoted respectively by the new numerators. Lastly, apply to each of the results so found unity divided by the common denominator, as index.*

Exercises. In each of the following exercises, reduce the several quantities to equivalent ones having the same index.

18. $3^{\frac{2}{3}}$ and $4^{\frac{1}{2}}$. *Ans.* $81^{\frac{1}{6}}$, and $64^{\frac{1}{6}}$.

19. $2^{\frac{1}{2}}$, $5^{\frac{1}{4}}$, and $7^{\frac{1}{8}}$. *Ans.* $16^{\frac{1}{8}}$, $25^{\frac{1}{8}}$, and $7^{\frac{1}{8}}$.

20. $x^{\frac{2}{3}}$, $y^{\frac{3}{4}}$, and $z^{\frac{5}{6}}$. *Ans.* $(x^8)^{\frac{1}{12}}$, $(y^9)^{\frac{1}{12}}$, and $(z^{10})^{\frac{1}{12}}$.

21. a, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, and $a^{\frac{1}{4}}$. *Ans.* $(a^{12})^{\frac{1}{12}}$, $(a^6)^{\frac{1}{12}}$, $(a^4)^{\frac{1}{12}}$, and $(a^3)^{\frac{1}{12}}$.

105. *The multiplication and division of radicals* are conducted on the same principles as the multiplication and division of integral quantities.

Thus, the product of $2a^{\frac{1}{2}}$ and $5a^{\frac{1}{3}}$ is $10a^{\frac{5}{6}}$. This is found according to §§ 37. and 38., by taking the product of the coefficients and adding the indices. To prepare for the addition of the indices, they must be reduced to equivalent ones having a common denominator. We thus get $\frac{3}{6}$ and $\frac{2}{6}$, the sum of which is $\frac{5}{6}$.

If, again, we divide $6x^{\frac{1}{3}}$ by $2x^{\frac{1}{4}}$, we get (§ 45.) $3x^{\frac{1}{3}-\frac{1}{4}}$, or $3x^{\frac{1}{12}}$. So likewise, $2x^{\frac{1}{3}} \div x^{\frac{1}{2}} = 2x^{\frac{1}{3}-\frac{1}{2}} = 2x^{-\frac{1}{6}} = \frac{2}{x^{\frac{1}{6}}}$.

106. *When two or more radicals have the same index*, their product will be obtained by finding the product of the quantities without the indices, and attaching to it the common index.

Thus, the product of $a^{\frac{1}{2}}$ and $(a+b)^{\frac{1}{2}}$ is $(a^2+ab)^{\frac{1}{2}}$; while that of $(a^2-ax+x^2)^{\frac{1}{3}}$ and $(a+x)^{\frac{1}{3}}$ is $(a^3+x^3)^{\frac{1}{3}}$.

On the same principle, if $(a^2-x^2)^{\frac{1}{2}}$ be divided by $(a-x)^{\frac{1}{2}}$, the quotient (§ 57.) is $(a+x)^{\frac{1}{2}}$.

The following additional examples will farther illustrate the subject.

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* In this way we may determine the comparative magnitudes of numbers affected by different indices, without performing the actual extraction. Thus, if the square root of 5 and the cube root of 11 be proposed, we have $5^{\frac{1}{2}}=5^{\frac{3}{6}}=125^{\frac{1}{6}}$, and $11^{\frac{1}{3}}=11^{\frac{2}{6}}=121^{\frac{1}{6}}$; the former of which is the greater. In a similar manner it would be shown, that the fourth root of 34 is less than the third root of 14.

Exam. 4.

$$\begin{array}{r}
3a^{\frac{2}{3}}-4a^{\frac{1}{3}}-2 \\
2a+4a^{\frac{2}{3}}-a^{\frac{1}{3}} \\
\hline
6a^{\frac{5}{3}}-8a^{\frac{4}{3}}-4a \qquad\qquad\qquad \\
12a^{\frac{4}{3}}-16a-8a^{\frac{2}{3}} \qquad \\
-3a+4a^{\frac{2}{3}}+2a^{\frac{1}{3}} \\
\hline
6a^{\frac{5}{3}}+4a^{\frac{4}{3}}-23a-4a^{\frac{2}{3}}+2a^{\frac{1}{3}}
\end{array}$$

Or thus:

$$\begin{array}{rrrrr}
3 & -4 & -2 & & \\
2 & 4 & -1 & & \\
\hline
6 & -8 & -4 & & \\
 & 12 & -16 & -8 & \\
 & & -3 & 4 & 2 \\
\hline
6 & 4 & -23 & -4 & 2
\end{array}$$

In this example, as the indices of a diminish from term to term by the same quantity, $\frac{1}{3}$, the method of detached coefficients is admissible, and its facility is strikingly felt.

Exam. 5. Divide $2x^{\frac{5}{4}}-3x+6x^{\frac{1}{2}}-8x^{\frac{1}{4}}$ by $x^{\frac{1}{2}}-2$

$$\begin{array}{l}
x^{\frac{1}{2}}-2)2x^{\frac{5}{4}}-3x+6x^{\frac{1}{2}}-8x^{\frac{1}{4}}(2x^{\frac{3}{4}}-3x^{\frac{1}{2}}+4x^{\frac{1}{4}} \\
\qquad\quad \underline{2x^{\frac{5}{4}}-4x^{\frac{3}{4}}} \\
\qquad\qquad\quad -3x+4x^{\frac{3}{4}}+6x^{\frac{1}{2}} \\
\qquad\qquad\quad \underline{-3x+6x^{\frac{1}{2}}} \\
\qquad\qquad\qquad\qquad 4x^{\frac{3}{4}}-8x^{\frac{1}{4}} \\
\qquad\qquad\qquad\qquad \underline{4x^{\frac{3}{4}}-8x^{\frac{1}{4}}} \\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

Or thus:

$$\begin{array}{rrrrr|rrr}
2 & -3 & 0 & 6 & -8 & 1 & 0 & -2 \\
\cline{6-8}
2 & 0 & -4 & & & 2 & -3 & 4 \\
\hline
 & -3 & 4 & 6 & & & & \\
 & -3 & 0 & 6 & & & & \\
\hline
 & & 4 & 0 & -8 & & & \\
 & & 4 & 0 & -8 & & & \\
\hline
 & & & & 0 & & &
\end{array}$$

This example also affords an instance of the advantage of employing only the coefficients. The process might have been abridged still more, as in Exam. 10. p. 37.

Exercises.

Find the products of the following quantities.

22. $2a^{\frac{1}{2}}$ and $3a^{\frac{1}{4}}$. *Ans.* $6a^{\frac{3}{4}}$.

23. $x^{\frac{2}{3}}$ and $4x^{\frac{3}{4}}$. *Ans.* $4x^{\frac{17}{12}}$.

24. $\sqrt{(6a)}$ and $\sqrt{(10a)}$. *Ans.* $2a\sqrt{15}$.

25. $\sqrt{(x^2-ax+a^2)}$ and $\sqrt{(x^2+ax+a^2)}$. *Ans.* $\sqrt{(x^4+a^2x^2+a^4)}$

26. $\sqrt{2}+1$ and $\sqrt{2}-1$. *Ans.* 1.

27. x, $x^{\frac{1}{2}}$, and $x^{\frac{1}{3}}$. *Ans.* $x^{\frac{11}{6}}$.

28. $\sqrt[3]{(12a^2)}$ and $\sqrt[3]{(14a^5)}$. *Ans.* $2a^2\sqrt[3]{(21a)}$.

29. $\sqrt[3]{(2x-1)}$, $\sqrt[3]{(2x+3)}$, and $\sqrt[3]{(x-1)}$.
Ans. $\sqrt[3]{(4x^3-7x+3)}$.

30. $\sqrt{(x+a)}$, and $\sqrt{(x-a)}$. *Ans.* $\sqrt{(x^2-a^2)}$.

31. $\sqrt{(x+2)}+1$ and $\sqrt{(x+2)}-1$. *Ans.* $x+1$.

32. $x-2x^{\frac{1}{2}}+3$ and $x^2+4x^{\frac{3}{2}}+6x$. *Ans.* $x^3+2x^{\frac{5}{2}}+x^2+18x$.

33. $2x^{\frac{3}{2}}-2x^{\frac{1}{2}}-\frac{1}{x^{\frac{1}{2}}}+\frac{3}{x^{\frac{3}{2}}}$ and $3x^{\frac{3}{2}}-\frac{2}{x^{\frac{1}{2}}}$.

Ans. $6x^3-6x^2-7x+13+\frac{2}{x}-\frac{6}{x^2}$.*

In the following exercises, divide the first quantity in each by the second.

34. $6a^{\frac{4}{5}}$ and $8a^{\frac{3}{10}}$ *Ans.* $\frac{3}{4}a^{\frac{1}{2}}$.

35. $\frac{4}{\sqrt{x}}$ and $3\sqrt{x}$. *Ans.* $\frac{4}{3x}$.

36. $\sqrt{\frac{x+1}{x-1}}$ and $\sqrt{\frac{x-1}{x+1}}$. *Ans.* $\frac{x+1}{x-1}$.

37. $\sqrt{30}$ and $\sqrt{5}$. *Ans.* $\sqrt{6}$.

38. $\sqrt[3]{(x^2y)}$ and $\sqrt[3]{(xy^2)}$. *Ans.* $\sqrt[3]{\frac{x}{y}}$.

39. $(x^4-3x^3-2x^2)^{\frac{1}{4}}$ and $x^{\frac{1}{2}}$. *Ans.* $(x^2-3x-2)^{\frac{1}{4}}$.

40. $\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}$ and $\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}}$. *Ans.* $\frac{x-2\sqrt{(ax)}+a}{x+2\sqrt{(ax)}+a}$.

41. $4x^{\frac{9}{5}}-9x^{\frac{7}{5}}+14x-19x^{\frac{3}{5}}+4x^{\frac{1}{5}}$ and $x^{\frac{6}{5}}-2x^{\frac{4}{5}}+3x^{\frac{2}{5}}-4$.

Ans. $4x^{\frac{3}{5}}-x^{\frac{1}{5}}$.

42. $x^{\frac{1}{2}}-a^{\frac{1}{2}}$ and $x^{\frac{1}{6}}-a^{\frac{1}{6}}$. *Ans.* $x^{\frac{1}{3}}+a^{\frac{1}{6}}x^{\frac{1}{6}}+a^{\frac{1}{3}}$.

43. $\frac{a-\sqrt{(a^2-b^2)}}{\sqrt{(a^2+b^2)}+b}$ and $\frac{\sqrt{(a^2+b^2)}-b}{a+\sqrt{(a^2-b^2)}}$. *Ans.* $\frac{b^2}{a^2}$.

107. If the numerator and denominator of a fractional radical be multiplied by a quantity, which will give a rational denominator in the result, the quantity so obtained will be simpler in its nature than the original one, especially, in reference to any numerical computations to which it may be subjected.

Thus, if we wish to extract the square root of $\frac{2}{3}$; that is, to find the value of $(\frac{2}{3})^{\frac{1}{2}}$, numerically, by multiplying the numerator and denominator by $3^{\frac{1}{2}}$, we get $\frac{6^{\frac{1}{2}}}{3}$. Hence the required root is one third of the square root of 6.†

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* This exercise will be most simply wrought by writing the given terms thus; $2x^{\frac{3}{2}}-2x^{\frac{1}{2}}-x^{-\frac{1}{2}}+3x^{-\frac{3}{2}}$, and $3x^{\frac{3}{2}}+0x^{\frac{1}{2}}-2x^{-\frac{1}{2}}$ and then proceeding as in the margin, by the method of detached coefficients.

$$\begin{array}{rrrrrr} 2 & -2 & -1 & 3 & & \\ 3 & 0 & -2 & & & \\ \hline 6 & -6 & -3 & 9 & & \\ & & -4 & 4 & 2 & -6 \\ \hline 6 & -6 & -7 & 13 & 2 & -6 \end{array}$$

† In this instance, the required root will be found either by actually extracting the square root of 6, or by taking it from the table in p. 80.; and the same plan may be followed in other similar cases.

In like manner, if we multiply the numerator and denominator of $(\frac{4}{7})^{\frac{1}{3}}$ by $7^{\frac{2}{3}}$, we get $\frac{(4\times 7^2)^{\frac{1}{3}}}{7}=\frac{(4\times 49)^{\frac{1}{3}}}{7}=\frac{196^{\frac{1}{3}}}{7}$. Hence the cube root of $\frac{4}{7}$ is one seventh of the cube root of 196.

In such cases, when the denominator is a monomial, a, with an index, $\frac{m}{n}$, the fraction will be changed into an equivalent one with a rational denominator, by multiplying its terms by a with the index $1-\frac{m}{n}$, or $\frac{n-m}{n}$.

108. If the denominator be a binomial of the form $a\pm b^{\frac{1}{2}}$, or $a^{\frac{1}{2}}\pm b^{\frac{1}{2}}$, the multiplier is the same as the denominator with one of its signs changed; that is, it is in the one case $a\mp b^{\frac{1}{2}}$, and in the other $a^{\frac{1}{2}}\mp b^{\frac{1}{2}}$.

Thus, by multiplying the terms of $\frac{3+\sqrt{7}}{3-\sqrt{7}}$ by $3+\sqrt{7}$, we get $\frac{3+\sqrt{7}}{3-\sqrt{7}}=8+3\sqrt{7}$: and from $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, by multiplying its terms by $\sqrt{7}+\sqrt{3}$, we get $\frac{7+2\sqrt{21}+3}{7-3}=\frac{10+2\sqrt{21}}{4}=\frac{1}{2}(5+\sqrt{21})$.

109. If the denominator be of the form $a^{\frac{1}{3}}\pm b^{\frac{1}{3}}$, the multiplier will be $a^{\frac{2}{3}}\mp a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$: and a general multiplier may be easily found, but it is seldom of use.*

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* Thus, by dividing every index by n in the general formula at the end of § 58., we get

$$\frac{x-a}{x^{\frac{1}{n}}-a^{\frac{1}{n}}}=x^{\frac{n-1}{n}}+a^{\frac{1}{n}}x^{\frac{n-2}{n}}+a^{\frac{2}{n}}x^{\frac{n-3}{n}}+,\ \&c.$$

where the second member is the multiplier. When both terms of the denominator are positive, the multiplier will be the same, except that the signs of its even terms (the second, fourth, &c.) are negative; as is plain from § 60. That the indices may be divided by n will readily appear by substituting throughout x' and a' for x^n and a^n, and dropping the accents in the result.

The reader cannot fail to observe, besides other advantages, how much the reductions pointed out above will facilitate computations regarding many radicals. Thus, in the last example in § 108., without the reduction, the square roots of 7 and 3 must be extracted to as many places of decimals as may be considered necessary; then their sum and difference must be taken, and the former must be divided by the latter. In the other mode, we have merely to extract one root, that of 21, to add 5 to it, and to halve the result.

Exercises. Express the following fractions with rational denominators.

44. $\sqrt{\dfrac{2}{11}}$. *Ans.* $\dfrac{\sqrt{22}}{11}$.

45. $\sqrt[3]{\dfrac{2}{11}}$. *Ans.* $\dfrac{\sqrt[3]{242}}{11}$.

46. $\dfrac{1}{\sqrt{2}+1}$. *Ans.* $\sqrt{2}-1$.

47. $\dfrac{1}{\sqrt{2}-1}$. *Ans.* $\sqrt{2}+1$.

48. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$. *Ans.* $3+2\sqrt{2}$.

49. $\dfrac{\sqrt{12}-\sqrt{10}}{\sqrt{6}+\sqrt{5}}$. *Ans.* $11\sqrt{2}-4\sqrt{15}$.

50. $\dfrac{\sqrt{(x+a)}+\sqrt{(x-a)}}{\sqrt{(x+a)}-\sqrt{(x-a)}}$. *Ans.* $\dfrac{x+\sqrt{(x^2-a^2)}}{a}$.

51. $\dfrac{1}{a\sqrt{x}+b\sqrt{y}}$. *Ans.* $\dfrac{a\sqrt{x}-b\sqrt{y}}{a^2x-b^2y}$.

52. and 53. Prove that $\dfrac{2x^2-x\sqrt{(x^2-a^2)}-a^2}{x-\sqrt{(x^2-a^2)}}=\dfrac{x^3+(x^2-a^2)^{\frac{3}{2}}}{a^2}$:

and that

$$\frac{1}{\sqrt{5}-\sqrt{10}+\sqrt{20}-\sqrt{40}+\sqrt{80}}=\frac{7\sqrt{5}+3\sqrt{10}}{155}.$$

54. Show that $\dfrac{a\sqrt{(a+x)}}{\sqrt{(a+x)}-\sqrt{x}}=a+x+\sqrt{(ax+x^2)}$.

Miscellaneous Exercises regarding Radicals.

55. Required the sum and difference of the squares of $\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$ and $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$. *Ans.* $\dfrac{2x^2+12x+2}{(x-1)^2}$, and $\dfrac{8(x+1)\sqrt{x}}{(x-1)^2}$.

56. In the last change x into xy^{-1}, and thus show, that the sum and difference of $\left(\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}\right)^2$ and $\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^2$ are $\dfrac{2x^2+12xy+2y^2}{(x-y)^2}$ and $\dfrac{8(x+y)x^{\frac{1}{2}}y^{\frac{1}{2}}}{(x-y)^2}$.

57. Prove that the product of $\dfrac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{3}}-y^{\frac{1}{3}}}$ and $\dfrac{x^{\frac{2}{3}}-y^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}$ is $1+\dfrac{2x^{\frac{1}{3}}y^{\frac{1}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}$.

58. Multiply $3x^{\frac{1}{12}}-4-\dfrac{2}{x^{\frac{1}{24}}}+\dfrac{5}{x^{\frac{1}{12}}}$ by $4x^{\frac{1}{12}}-1+\dfrac{3}{x^{\frac{1}{24}}}-\dfrac{2}{x^{\frac{1}{8}}}$, by the method of detached coefficients.*

Ans. $12x^{\frac{4}{24}}-19x^{\frac{2}{24}}+x^{\frac{1}{24}}+24-16x^{-\frac{1}{24}}-11x^{-\frac{2}{24}}+23x^{-\frac{3}{24}}+4x^{-\frac{4}{24}}-10x^{-\frac{5}{24}}$.

59. Divide $3x+6x^{\frac{7}{6}}-x^{\frac{5}{4}}+11x^{\frac{4}{3}}-2x^{\frac{17}{12}}+4x^{\frac{3}{2}}-3x^{\frac{19}{12}}+6x^{\frac{5}{3}}$ by $3x^{\frac{1}{2}}-x^{\frac{3}{4}}+2x^{\frac{5}{6}}$, using only the coefficients. *Ans.* $x^{\frac{1}{2}}+2x^{\frac{2}{3}}+3x^{\frac{5}{6}}$.

60. Show that $\{[(x^{-1})^{-2}]^{-3}\}^{-4}=x^{24}$; and
that $\{[(x^{-\frac{1}{2}})^{-\frac{1}{3}}]^{-\frac{1}{4}}\}^{-\frac{1}{5}}=x^{\frac{1}{120}}$.

61. Prove that

$$\left(\sqrt{\frac{1+x}{1-x}}-\sqrt{\frac{1-x}{1+x}}\right)\div\left(\sqrt{\frac{1+x}{1-x}}+\sqrt{\frac{1-x}{1+x}}\right)=x.$$

62. Reduce $\dfrac{a-x}{a^{\frac{5}{2}}-a^{2}x^{\frac{1}{2}}+2a^{\frac{3}{2}}x-2ax^{\frac{3}{2}}+a^{\frac{1}{2}}x^{2}-x^{\frac{5}{2}}}$ to its lowest terms.

Ans. $\dfrac{a^{\frac{1}{2}}+x^{\frac{1}{2}}}{(a+x)^{2}}$.

110. The principal elementary processes in the resolution of equations have been given in Chapter I. It still remains, however, that we should investigate the method of managing equations in which the unknown quantity is involved in terms which are radicals. This we are now prepared to do; and the necessary reductions will be effected by means of one or other of the following methods.

When an equation contains a single radical, let that radical, by

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* Here (*Arith.* p. 86.) the indices of the given factors become respectively, by reducing them to equivalent ones having a common denominator, $\frac{2}{24}$, 0, $-\frac{1}{24}$, $-\frac{2}{24}$; and $\frac{2}{24}$, 0, $-\frac{1}{24}$, $-\frac{3}{24}$. Now, in both of these, the index $\frac{1}{24}$ is wanting; as is also $-\frac{2}{24}$ in the second. We put 0, therefore, as the coefficient of each of the terms in which these quantities would occur, and the detached coefficients stand as in the margin. The work then proceeds in the usual way. In the answer, for rendering the subject plainer to the learner, the indices are left unreduced, and the quantities having negative indices are not taken to the denominator. The learner ought to make those changes.

$$\begin{array}{lllll} 3 & 0-4 & -\,2 & 5 & \\ 4 & 0-1 & 3 & 0-2 & \end{array}$$

transposition, or other operations, be made to stand alone, if it do not stand so already. Then, if both members be raised to the power corresponding to the root expressed in the radical, a new equation will be obtained, which will be free from radicals.

Exam. 6. Given the equation, $x-\sqrt{(x^2-5)}=1$, to find x.

Here, we get $x-1=\sqrt{(x^2-5)}$, by transposition; and thence, by squaring both members, $x^2-2x+1=x^2-5$. Then, by rejecting x^2, and resolving the equation, we get $x=3$.*

Exam. 7. Resolve the equation, $\sqrt{\frac{x+1}{x-7}}=3$.

Here, by squaring and clearing of fractions, we get $x+1=9x-63$; whence $x=8$.

Exam. 8. Resolve the equation, $a+b\sqrt[n]{(x-c)}=d$.

Here, we get $x=c+\left(\frac{d-a}{b}\right)^n$, by transposing a, dividing by b, raising the members of the result to their nth powers, and transposing $-c$.

Exercises.

Resolve the following equations.

63. $x-3+\sqrt{(4x^2-3x-4)}=3x-4$. *Ans.* $x=5$

64. $\sqrt{(b^2x^2+c^2)}+bx=d$. *Ans.* $x=\frac{d^2-c^2}{2bd}$.

65. $x+a+\sqrt{(x^2+2ax+b^2)}=c$.

Ans. $x=\frac{a^2-b^2+c^2}{2c}-a=\frac{(a-c)^2-b^2}{2c}$.

111. ***When an equation contains more radicals than one***, we may remove one of them as in the last §; then, after modifying the result, we may remove another by the same means; and thus we may proceed till none remains.

Exam. 9. Resolve the equation, $\sqrt{x}+\sqrt{(x+7)}=7$.

Here, by transposing $\sqrt{x}$, and by squaring the members of the result, we get $x+7=49-14\sqrt{x}+x$. Hence, by rejecting x, transposing $14x$ and 7, and dividing by 14, we obtain $\sqrt{x}=3$, and therefore $x=9$.

---

* Had the proposed equation been $x+\sqrt{(x^2-5)}=1$, we should still have found $x=3$. In this case -3 must be taken as the root of the radical; so that in reality the equation in this form is identical with the one in the text.

Exam. 10. To generalise the last example, let it be required to resolve the equation, $\sqrt{x}+\sqrt{(x+a)}=b$.

Here, by working as in the last example, we get $x=\left(\frac{b^2-a}{2b}\right)^2$. If we had transposed $\sqrt{(x+a)}$ instead of $\sqrt{x}$, squared, &c., we should have got $x=\left(\frac{b^2+a}{2b}\right)^2-a$, a value which is easily reduced to the same form as the last.

112. In some cases, the solution may be obtained neatly and simply, without separating the radicals. This will often be so, when the radicals bear any peculiar relations to each other. In particular instances also, methods may be employed, for which it would be difficult to give a general rule.

Exam. 11. Resolve the equation, $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$.

Here, by cubing both members according to § 56., we get $a+x+a-x+3b\sqrt[3]{(a^2-x^2)}=b^3$. Hence, by contracting, transposing $2a$, and dividing by $3b$, we obtain $\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{3b}$.

By cubing this, we get $a^2-x^2=\left(\frac{b^3-2a}{3b}\right)^3$; whence, by transposition, and by extracting the square root, we find

$$x=\sqrt{\left\{a^2-\left(\frac{b^3-2a}{3b}\right)^3\right\}}.$$

Exam. 12. Resolve the equation, $\sqrt{ax}=\sqrt{bx}+\sqrt{c}$.

Here, we obtain by transposition, $\sqrt{ax}-\sqrt{bx}=\sqrt{c}$, or, § 51. $(\sqrt{a}-\sqrt{b})\sqrt{x}=\sqrt{c}$. Hence, by dividing by $\sqrt{a}-\sqrt{b}$, and squaring both members of the result, we get $x=\frac{c}{(\sqrt{a}-\sqrt{b})^2}$.

Exercises in the Resolution of Equations containing Radicals.

Resolve the following equations.

66. $\sqrt{(x+7)}+3=7$. *Ans.* $x=9$.

67. $\sqrt{(x+a)}+b=c$. *Ans.* $x=-a+(c-b)^2=-a+(b-c)^2$.

68. $x+\sqrt{(x^2+11)}=11$. *Ans.* $x=5$.

69. $x+\sqrt{(x^2+a)}=b$. *Ans.* $x=\frac{b^2-a}{2b}$.

70. $a\sqrt{x}+b\sqrt{x}-c\sqrt{x}=d.$ *Ans.* $x=\left(\frac{d}{a+b-c}\right)^2$

71. $\sqrt{(2x-3a)}+\sqrt{2x}=3\sqrt{a}.$ *Ans.* $x=2a.$

72. $\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{a}}=\sqrt{\frac{b}{a}}.$ *Ans.* $x=\left(\frac{-2\sqrt{ab}}{\sqrt{a}-\sqrt{b}}\right)^2.$

73. $\frac{4a+\sqrt{x}}{3b+\sqrt{x}}=\frac{2a+\sqrt{x}}{b+\sqrt{x}}.$ *Ans.* $x=\left(\frac{ab}{a-b}\right)^2.$

74. $\sqrt{(4a+x)}=2\sqrt{(b+x)}-\sqrt{x}.$ *Ans.* $x=\frac{(b-a)^2}{2a-b}.$

75. $\sqrt{\{13+\sqrt{[7+\sqrt{(3+\sqrt{x})}]}\}}=4.$ *Ans.* $x=1.$

76. $\sqrt{\frac{x+a}{x-a}}+\sqrt{\frac{x-a}{x+a}}=b.$ *Ans.* $x=\frac{ab}{\sqrt{(b^2-4)}}.$

77. $\sqrt{(x^2+ax+a^2)}+\sqrt{(x^2-ax+a^2)}=\sqrt{(2a^2-2b^2)}.$

Ans. $x=\sqrt{\frac{b^4-a^4}{a^2-2b^2}}.$

CHAPTER VI.

INVOLUTION, EVOLUTION, IMAGINARY QUANTITIES, ETC.

113. When we find any assigned power of a given quantity, the process is termed *involution.* The converse process, the finding, or, as it is generally expressed, the extracting, of an assigned root of a given quantity, is called *evolution.**

114. Let t_1, t_2, t_3, &c., be the successive terms of a compound quantity. Then, according to the method pointed out in § 54., we have

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* It is evident from § 10., that a power, in its primitive meaning, may always be found by means of multiplication; and we have already had several instances of the finding of powers and the extracting of roots. In this Chapter, evolution will be taken up more particularly, and in more detail; and in a subsequent part of the work the consideration of both involution and evolution will be resumed, in establishing and applying the binomial theorem.

$$(t_1+t_2)^2=t_1^2+2t_1t_2+t_2^2;$$
$$(t_1+t_2+t_3)^2=t_1^2+2t_1t_2+t_2^2+2(t_1+t_2)t_3+t_3^2;$$
$$(t_1+t_2+t_3+t_4)^2=t_1^2+2t_1t_2+t_2^2+2(t_1+t_2)t_3+t_3^2$$
$$+2(t_1+t_2+t_3)t_4+t_4^2;$$

or, as the last may be written,

$$\begin{aligned}(t_1+t_2+t_3+t_4)^2= {} & t_1^2\\ & +(2\,t_1+t_2)t_2\\ & +\{2(t_1+t_2)+t_3\}t_3\\ & +\{2(t_1+t_2+t_3)+t_4\}t_4.\end{aligned}$$

In this arrangement, the first line of the second member is the square of t_1, the first term of the polynomial; the first and second lines together are the square of t_1+t_2; the first three lines are the square of $t_1+t_2+t_3$; and the whole four the square of $t_1+t_2+t_3+t_4$. We see also that the second line is the product of the second term into the sum of that term and twice the first; that the third line is the product of the third term into the sum of that term and twice the two preceding it; and that the fourth is the product of the fourth term into the sum of that term and the double of all the terms that precede it. By examining the mode of formation of these quantities, we shall see that if there were additional terms, such as t_5, t_6, &c.; and if the foregoing arrangement were continued, the same property would hold universally, the *n*th line being the product of the *n*th term into the sum of that term and the double of all the preceding ones.

115. From the relations that have now been arrived at, we are enabled to establish the method of extracting the square root of a compound quantity, such as the second member of the equation in § 114. Thus, the square root of the first line gives t_1. Subtracting the square of this from the second member, we have remaining the second, third, and fourth lines. We then see, that t_2 will be found by dividing the first of the remaining terms, $2t_1t_2$, by $2t_1$, the double of the part already obtained. Adding t_2, when so found, to the divisor $2t_1$, and multiplying the sum by t_2, we get $(2t_1+t_2)t_2$; and, as this is the same as the second line, by subtracting it from the three lines which we had formerly remaining, we leave simply the third and fourth. In like manner, t_3 will be found by dividing the first term $(2t_1t_3)$ of the third line by $2t_1$. Then, by adding t_3 to $2\,(t_1+t_2)$, and multiplying the sum by t_3, we get the third line and taking this from the last remainder, the third and fourth

lines, we have now only the fourth line remaining. By dividing the first term of this remainder by $2t_1$, we get t_4. Then, by adding this to $2(t_1+t_2+t_3)$, and multiplying the sum by t_4, we obtain for product the fourth line. Subtracting it, therefore, we have no remainder, and the work terminates: and it is plain, that this process will succeed in every case in which the proposed quantity is a complete square, such as $(t_1+t_2+\ldots\ldots+t_n)^2$. When the given quantity is not a complete square, the root may be carried out in a series to any extent we please, by a continuation of the same process, as is done in the analogous case in common arithmetic by means of decimal fractions.

It may be remarked that, as in multiplication and division, so in the extraction of roots, the quantities should be arranged (§ 39.) according to the powers of one of the quantities concerned. Any other arrangement would give irregular and complicated results that would be of no use.

The substance of the conclusions at which we have arrived, might be expressed in a formal rule. Their application, however, will perhaps be better understood from the following examples.

Exam. 1. Find the square root of $4x^4-24ax^3+108ax^3+81a^4$.

$$\begin{array}{rl}
 & 4x^4-24ax^3+108a^3x+81a^4\ (2x^2-6ax-9a^2 \\
 & \underline{4x^4} \\
4x^2-6ax) & -24ax^3 \\
\underline{-6ax} & \underline{-24ax^3+36a^2x^2} \\
4x^2-12ax-9a^2) & \quad -36a^2x^2+108a^3x+81a^4 \\
 & \quad \underline{-36a^2x^2+108a^3x+81a^4} \\
 & \qquad\qquad 0
\end{array}$$

Or thus:

$$\begin{array}{rrrrrrl}
 & 4 & -24 & 0 & 108 & 81 & (2\quad -6\quad -9,\quad \text{or} \\
 & \underline{4} & & & & & \;\; 2x^2-6ax-9a^2. \\
4-6) & & -24 & 0 & & & \\
\underline{-6} & & -24 & \underline{36} & & & \\
4-12-9) & & & -36 & 108 & 81 & \\
 & & & \underline{-36} & \underline{108} & \underline{81} & \\
 & & & & 0 & &
\end{array}$$

In the foregoing work, which is arranged according to the descending powers of x, and the ascending ones of a, we have $2x^2$ for the first term of the root; the subtraction of the square of which destroys the first of the given terms. In division, we bring down, each time, one term. In extracting the square root, however, the divisor is each time increased by a term, and therefore we annex two terms to each remainder. In the present example it is unnecessary to take down two terms, as the latter of the terms that would be brought down, a term containing x^2, is wanting. We take down, therefore, $-24ax^3$; and dividing this by $4x^2$, the double of what has been already found, we get $-6ax$, which is written both as a part of the root and a part of the divisor. Then, multiplying the divisor by $-6ax$, and subtracting as in division, we get $-36a^2x^2$; to which we annex the two remaining terms. We add also $-6ax$, the term last found, to the last complete divisor; and thus we get, in an easy way, the double of the part of the root already found. By dividing $-36a^2x^2$ by $4x^2$, we get $-9a^2$, which is annexed both to the root and the divisor. Then, by multiplying and subtracting, as in division, we find no remainder; and therefore $2x^2-6ax-9a^2$ is the root. The work, by means of detached coefficients, is also given, but without any attempt at contraction, except the omission of the powers of x and a.

Exam. 2. Find the square root of $x^4-4x^3+10x^2+9$.

$$\begin{array}{rll}
 & x^4-4x^3+10x^2+9 & (x^2-2x+3 \\
 & x^4 & \\
\hline
2x^2-2x) & -4x^3+10x^2 & \\
-2x & -4x^3+\ \ 4x^2 & \\
\hline
2x^2-4x+3\ ) & \qquad 6x^2+9 & \\
 & \qquad 6x^2-12x+9 & \\
\hline
 & \qquad\qquad 12x &
\end{array}$$

Or thus:

$$\begin{array}{rrrrrrrrrrrrr}
 & & & & 1 & -4 & 10 & 0 & 9\,(1 & -2 & 3 & 6 & 12 \\
 & & & & 1 & & & & & & & & \\
\hline
2 & -2) & & & & -4 & 10 & & & & & & \\
 & -2 & & & & -4 & 4 & & & & & & \\
\hline
2 & -4 & 3\ ) & & & & 6 & 0 & 9 & & & & \\
 & & 3 & & & & 6 & -12 & 9 & & & & \\
\hline
2 & -4 & 6 & 6\ ) & & & 12 & 0 & 0 & 0 & & & \\
 & & & 6 & & & 12 & -24 & 36 & 36 & & & \\
\hline
2 & -4 & 6 & 12 & &) & 24 & -36 & -36 & & & &
\end{array}$$

In this exercise, after we have obtained x^2-2x+3, there is the remainder $12x$; and, as in dividing this by $2x^2$, the first term of the divisor, the term so obtained would contain x in its denominator, according to the usual practice the work may be regarded as terminating here with the remainder $12x$, which, if we proved the operation by multiplying the root by itself, must be added to the product. The work might be carried farther, however, so as to give a series. This is most easily effected by means of detached coefficients, as in the second process, and the root is found to be $x^2-2x+3+6x^{-1}+12x^{-2}+$&c.

Exercises. Find the square roots of the following quantities.

1. x^2-6x+9.	*Ans.* $x-3$.
2. $4x^2+4ax+a^2$.	*Ans.* $2x+a$.
3. $x^2\pm 2+x^{-2}$.	*Ans.* $x\pm x^{-1}$.
4. $16x-24+9x^{-1}$.	*Ans.* $4x^{\frac{1}{2}}-3x^{-\frac{1}{2}}$.
5. x^4+4x^3-8x+4.	*Ans.* x^2+2x-2.
6. $x^6+8x^5-80x^3+128x+64$.	*Ans.* x^3+4x^2-8x-8.
7. $1\pm x$.	*Ans.* $1\pm\frac{x}{2}-\frac{x^2}{8}\pm\frac{x^3}{16}-\frac{5x^4}{128}\pm$&c.

116. The nth power of a polynomial $x+y+\ldots$, which has x for its first term, and y for its second, commences with the two terms, x^n and $nx^{n-1}y$. To prove this, when n is an integer, let us multiply as in the margin, so as to get the first and second terms for the second and third powers; a third term z being annexed, to show that it, or others that might follow it, would not affect the first and second terms of the resulting powers. In this way, t_3 and t'_3 being put, for brevity, to denote the third terms, we find, that the first and second terms, in the second power, are x^2+2xy, and, in the third power, x^3+3x^2y; which are of the form above stated. Now, we can show, that, if the property be true for any one power, it is also true for the one next above it. To prove this, let us *suppose* it true for the nth power, and let us multiply, as in the margin, by $x+y+$ &c., to find the next power. In this way, we get

$$\begin{array}{rl}
x+y+z & +\ \&c. \\
x+y+z & +\ \&c. \\
\hline
x^2+xy+xz & +\ \&c. \\
xy+y^2 & +\ \&c. \\
xz & +\ \&c. \\
\hline
x^2+2xy+t_3 & +\ \&c. \\
x+y+z & +\ \&c. \\
\hline
x^3+2x^2y+t_3x & +\ \&c. \\
x^2y+2xy^2 & +\ \&c. \\
x^2z & +\ \&c. \\
\hline
x^3+3x^2y+t'_3 & +\ \&c.
\end{array}$$

$x^{n+1}+(n+1)\,x^n y+$ &c.: in which the property holds, the index of x in the first term, and the coefficient in the second, being each $n+1$, which denotes the order of the power; while in the second term, y is multiplied by the next lower power of x.

$$\begin{array}{llll}
x^n & +nx^{n-1}y & & +\ \text{\&c.} \\
x & +\ y & & +\ \text{\&c.} \\
\hline
x^{n+1} & +nx^n y & & +\ \text{\&c.} \\
 & \quad x^n y & & +\ \text{\&c.} \\
\hline
x^{n+1} & +(n+1)x^n y & + & \text{\&c.}
\end{array}$$

Now it has been shown that the property holds in the second and third powers; it must therefore hold in the next higher, the fourth; and, holding in the fourth, it must hold in the fifth, and therefore in the sixth; and so on: so that it is true universally.

117. This property, it is plain, will be true equally in reference to the powers of a binomial $x+y$; and it will appear hereafter that it also holds when n is negative or fractional. What has now been shown, however, is sufficient for our present purpose, which is to establish the following *general rule for extracting the roots of compound quantities.*

To find the n*th root of a compound quantity,* arrange the terms as in § 39.; then the nth root of the first term will be the first term of the required root. Take the nth power of this term from the given quantity, and divide the first term of the remainder by n times the term already found, raised to the power $n-1$; the quotient will be the next term of the root. Annex this to the term already found; raise the binomial so obtained to the nth power, and subtract the result from the given quantity. If there be no remainder, the operation is terminated; but, if there be a remainder, divide its first term by the same divisor as before, to get the next term of the root. Involve the trinomial thus found to the nth power, and subtract as before: and thus proceed till there is no remainder, or till a sufficient number of terms has been obtained. The reason of the process will be evident from the following example, in connexion with the principle established in § 116.

Exam. 3. Given $x^6+6ax^5-40a^3x^3+96a^5x-64a^6$; to find its third root.

$$\begin{array}{rl}
 & x^6+6ax^5-40a^3x^3+96a^5x-64a^6\,(x^2+2ax-4a^2 \\
 & x^6 \\
\cline{2-2}
3x^4) & \quad 6ax^5 \\
\cline{2-2}
 & x^6+6ax^5+12a^2x^4+8a^3x^3=(x^2+2ax)^3 \\
\cline{2-2}
3x^4) & \qquad -12a^2x^4 \\
\cline{2-2}
 & x^6+6ax^5-40a^3x^3+96a^5x-64a^6=(x^2+2ax-4a^2)^3. \\
\cline{2-2}
 & \qquad 0
\end{array}$$

Here the third root of x^6 is x^2, the cube of which being taken away, we get $6ax^5$, the first term of the remainder. Then, $n-1$, or $3-1$, being 2, we raise x^2 to the second power; the product of which, by $3(=n)$ is $3x^4$. Dividing $6ax^5$ by this, we obtain $2ax$. Hence we have x^2+2ax as part of the root. By finding the third power of this, and subtracting it from the given quantity, we find that the first term of the remainder is $-12a^2x^4$; the division of which by $3x^4$ gives $-4a^2$, another term of the root. This completes the root, as the third power of $x^2+2ax-4a^2$ is found to be the same as the given quantity.*

In operations of this kind, the method of detached coefficients may be used with much advantage.

Exercises. Extract the third root in the first of the following exercises, and the fifth in the second.

8. $64x^6-288x^5+1080x^3-1458x-729$. *Ans.* $4x^2-6x-9$.

9. $243x^5-810ax^4+1080a^2x^3-720a^3x^2+240a^4x-32a^5$. *Ans.* $3x-2a$.

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* The method of extracting roots explained above, is generally laborious, and is seldom required in practice. It may be remarked also, that in many instances roots may be readily discovered by inspection, and sometimes by other expedients. Thus, if it were required to find the fourth root of $x^4-8x^3+24x^2-32x+16$, the fourth roots of the first and last terms being x and 2, and the signs in the given quantity being partly positive and partly negative, we try $x-2$ as the root; and by raising it to the fourth power, we find it to be correct, as the result is equal to the given quantity. In Exam. 3., also, the third roots of the first and last terms are x^2 and $-4a^2$, cubing, therefore, x^2-4a^2, we get $x^6-12a^2x^4+48a^4x^2-64a^6$; and subtracting this from the given quantity, we find that the first and last terms of the remainder are $6ax^5$ and $96a^5x$. Then if we divide the first of these by three times the square of the first term of x^2-4a^2, and the last by three times the square of its latter part, we get the same quotient, $2ax$: which, therefore, we may be sure is the remaining term, if there be an exact root. The reason of this will be seen by applying the principle established in § 116., to the given quantity, both as that quantity stands, and also in a reversed order.

It should also be recollected, that, as is apparent from § 95., the fourth root may be obtained by two extractions of the second root; the sixth by one extraction of the second, and one of the third, or by one of the third and one of the second, &c. It may be farther remarked, that algebraists have given direct and distinct rules for the extraction of the third and higher roots, derived from the form of the third and the succeeding powers of a binomial. These are easily investigated, but are of little practical value.

118. By the nature of powers, and by the rule of the signs in multiplication, every power of a positive quantity is positive. So also are the *even* powers of a negative quantity; but its odd powers are negative.

Thus, every power of x is positive: also $(-x)^2 = x^2$, $(-x)^4 = x^4$, &c.; while $(-x)^3 = -x^3$, $(-x)^5 = -x^5$, &c.; or, in general terms, $(+x)^n = x^n$; $(-x)^{2n} = x^{2n}$; and $(-x)^{2n+1} = -x^{2n+1}$; n being a whole number: or somewhat differently, $(+x)^n = x^n$, and $(-x)^n = (-1x)^n = (-1)^n x^n$.

119. Hence, since $(+x)^2 = x^2$, and $(-x)^2 = x^2$; or, conjointly, $(\pm x)^2 = x^2$; it follows, conversely, that $\sqrt{x^2} = x$, or $-x$, or as it may be expressed, $\sqrt{x^2} = \pm x$. Hence we see, that, in the algebraic sense, *every number or quantity has two square roots, which are equal in magnitude, but have opposite signs.*

Thus, the square root of 100 is either 10 or -10, since $10 \times 10 = 100$, and $-10 \times -10 = 100$. In like manner, the square root of $x^2 - 2ax + a^2$ is $x - a$, or $a - x$; while that of $x^2 + 2ax + a^2$ is $x + a$ or $-x - a$; and the answer to Exam. 1. p. 93. is either what is there given, or $9a^2 + 6ax - 2x^2$, which is obtained from the other by changing its signs. So likewise, 2 and -2 are fourth roots of 16, and are sixth roots of 64. We thus see also, that a quantity of the form, $a + \sqrt{b}$, has two values. Thus, $7 + \sqrt{9}$ is either 10 or 4, since the square root of 9 is either 3 or -3.

120. In extracting odd roots, the case is different; as we do not find two equal roots with opposite signs. Thus, for the third root of 27, we get 3, and not -3; as the cube of -3 is not 27, but -27, which is not the number proposed; while, for the fifth root of -32, we get, not 2, but -2, the fifth power of the latter, and not of the former, being -32.

121. It follows from § 119., that a negative quantity has no *even* root; since any quantity, either positive or negative, raised to an even power, gives a positive result. Thus, $-x^2$ has no square root; since the square, not only of x, but of $-x$, is x^2, and not $-x^2$. Hence, any such expression as $\sqrt{-x^2}$, indicates an operation which it is impossible to perform; and it is, therefore, called an *imaginary* or *impossible quantity.* We shall see in a subsequent part of the work, that, in many instances, when quantities of this kind occur in the solution of a problem, they serve the purpose of pointing out in what circumstances it is possible, and in what impossible. They are also of much use on other occasions, particularly as instruments of investigation.

As distinguished from imaginary quantities, others are called *real* quantities.

122. It is both usual and advantageous to express imaginary quantities of the second degree, so that they may have the symbol $\sqrt{-1}$, as a factor, which is always possible. Thus, since $-a^2=a^2\times-1$, we have (§ 98.) $\sqrt{-a^2}=a\sqrt{-1}$. In like manner, we should have $\sqrt{-b}$, or $\sqrt{(b\times-1)}=\sqrt{b}\times\sqrt{-1}$ or $=b'\sqrt{-1}$, if b' be put to denote $\sqrt{b}$. This mode of notation facilitates operations on imaginary quantities, and obviates difficulties that would otherwise be felt. With this notation, indeed, if we merely keep in mind, that, according to the nature of roots and powers, the square of $\sqrt{-1}$ is -1, we can perform on imaginary quantities, and by the same rules, all the operations to which real quantities can be subjected.

Thus, the product of $a\sqrt{-1}$ and $b\sqrt{-1}$ is $ab\times-1$ or $-ab$; and the square of $a\sqrt{-1}$ is $a^2\times-1$ or $-a^2$. If, again, $a\sqrt{-1}$ be divided by $b\sqrt{-1}$, the quotient is $\dfrac{a\sqrt{-1}}{b\sqrt{-1}}$, or simply $\dfrac{a}{b}$.

The example in the margin is interesting, as affording one instance, out of many, in which imaginary quantities used in operations disappear in the conclusions, and thus give real results. The product in this exercise might also be obtained by means of § 57., since the factors are the sum and difference of a and $b\sqrt{-1}$; the squares of which are a^2 and $-b^2$: and, by taking the latter from the former, we get a^2+b^2, as before. From this example, in connexion with § 57., we see how any binomial, having both its terms positive, may be exhibited as the product of two imaginary factors. Thus, $4a^2+9b^2$ may be written $4a^2-9b^2\times-1$. Then, the square roots of $4a^2$ and $9b^2\times-1$ being $2a$ and $3b\sqrt{-1}$, we get for the required factors, $2a+3b\sqrt{-1}$ and $2a-3b\sqrt{-1}$. In like manner, 2, or $1+1$, may be put under the form $(1+\sqrt{-1})(1-\sqrt{-1})$.

$$\begin{array}{r} a+b\sqrt{-1}\\ a-b\sqrt{-1}\\ \hline a^2+ab\sqrt{-1}\qquad\\ -ab\sqrt{-1}+b^2\\ \hline a^2+b^2\qquad\qquad\qquad \end{array}$$

123. The square of $\sqrt{-1}$ being -1, by multiplying this by $\sqrt{-1}$, we get, for the third power, $-\sqrt{-1}$; the product of which by $\sqrt{-1}$ (or the square of the second power, -1) is 1, the fourth power. The first four powers, therefore, of this symbol are $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1. Now, by multiplying the first

power, the second, the third, &c., by the fourth power, we get (§ 38.) the fifth power, the sixth, the seventh, &c.: and since the fourth power is 1, it follows that the fifth power, the sixth, the seventh, &c., are the same over again as the first, second, third, &c.; and, on the same principle, the ninth power, the tenth, &c. will be likewise the same. Hence if n be any whole number, we have

the 4th, 8th, 12th ... $4n$th powers of $\sqrt{-1}$ each $=1$;
the 1st, 5th, 9th ... $(4n+1)$th or $(4n-3)$th - $=\sqrt{-1}$;
the 2d, 6th, 10th ... $(4n\pm2)$th - - - $=-1$; and
the 3d, 7th, 11th ... $(4n+3)$th or $(4n-1)$th - $=-\sqrt{-1}$.

Exercises.

10. Multiply $2a+3b\sqrt{-1}$ by $3a-2b\sqrt{-1}$.
Ans. $6a^2+5ab\sqrt{-1}+6b^2$.

11. Cube $a+b\sqrt{-1}$. *Ans.* $a^3+3a^2b\sqrt{-1}-3ab^2-b^3\sqrt{-1}$.

12. Find the fourth powers of $1-2\sqrt{-1}$, and $2-\sqrt{-1}$.
Ans. $24\sqrt{-1}-7$, and $-24\sqrt{-1}-7$.

13. Cube $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$. *Ans.* 1.

14. Show that $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}=\sqrt{-1}$, and that $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}=\dfrac{a^2+2ab\sqrt{-1}-b^2}{a^2+b^2}$.

15. 25^2 is equal to 24^2+7^2, and also to 15^2+20^2. Hence resolve 25^2 into imaginary factors.
Ans. $24+7\sqrt{-1}$, and $24-7\sqrt{-1}$; or
$7+24\sqrt{-1}$, and $7-24\sqrt{-1}$:
also $15+20\sqrt{-1}$, and $15-20\sqrt{-1}$; or
$20+15\sqrt{-1}$, and $20-15\sqrt{-1}$.

124. The following principles, which will be useful in subsequent inquiries, and in the investigation of which we are assisted in part by what is established in this section, may be introduced here.

A quantity, however small, may be taken so many times, that the result shall exceed any quantity, however great.

To prove this, let a be any quantity, and b another exceeding it in any degree whatever. Then, let b be increased by c, and let it be assumed, that $na=b+c$. Hence, dividing by a we get $n=\dfrac{b+c}{a}$, a formula by means of which we can find a number n, such that if a be multiplied by it, the product shall be $b+c$; which is greater than b by c.

Thus, as an example, let $a=0{\cdot}001$ (the thousandth part of a unit), and $b=1000$, and we shall have $n=\frac{1000+c}{0{\cdot}001}$, or, by multiplying the numerator and denominator by 1000, $n=1000000+1000c$; and thus n is assigned so that, if 0·001 be multiplied by it, the product shall exceed 1000, being equal to $1000+c$.

125. The following principle is of equal importance. *If a quantity be greater than unity by any quantity, however small, a power of it may be found, which shall exceed any given quantity, however great.* To prove this, let a be a positive quantity, and consequently $1+a$ will be greater than unity. Then, by the rule for the signs in multiplication, all the terms of every power of $1+a$ will be positive; and, by § 114., the first and second terms of $(1+a)^n$ will be 1 and na. Hence, $(1+a)^n$ may be represented by $1+na+s$, where s is a positive quantity, being the sum of all the terms after $1+na$, which, as we have just seen, are all positive. Now, by § 124., n may be taken so great, that even the second term na alone would exceed any number, however great: much more, then, will $1+na+s$, that is, $(1+a)^n$, be greater than any assigned number.

126. It may be shown in the last place, that *if there be a constant quantity, a, a quantity, d, may be found, such that if a be divided by it, the quotient shall exceed any quantity, b, however great: and another quantity, d′, may be found, such that if a be divided by it, the quotient shall be less than any quantity, b′, however small.* To prove the first, we have merely to assume $\frac{a}{d}=b+c$, where c is any positive quantity; then, by multiplying by d, and dividing by $b+c$, we get $d=\frac{a}{b+c}$. To prove the second, assume $\frac{a}{d'}=b'-c'$, where c' is a positive quantity less than b': then $d'=\frac{a}{b'-c'}$. Hence we see, that to find the one divisor, d, we have merely to divide a by a quantity greater than b, by any quantity c, however small; while to find d', we have to divide the same by a quantity less than b' by a quantity, c', which may be as small as we please.

As an example, let it be required to find a divisor for 2, such

power, the second, the third,
(§ 38.) the fifth power, the
fourth power is 1, it follows
seventh, &c., are the same as
and, on the same principle, the
likewise the same. Hence if

the 4th, 8th, 12th ... 4nth
the 1st, 5th, 9th ... (4n+
the 2d, 6th, 10th ... (4n+
the 3d, 7th, 11th ... (4n+

Ex

10. Multiply $2a+3b\sqrt{-1}$ by

11. Cube $a+b\sqrt{-1}$. *Ans.* a^3

12. Find the fourth powers of 1−

Ans. 24√−

13. Cube $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2}-$

14. Show that $\frac{1+\sqrt{-1}}{1-\sqrt{-1}}=\sqrt{-1}$.

$\frac{a^2+2ab\sqrt{-1}-b^2}{a^2+b^2}$.

15. 25^2 is equal to 24^2+7^2, and also
resolve 25^2 into imaginary factors.

Ans. $24+7\sqrt{-1}$,
$7+24\sqrt{-1}$,
also $15+20\sqrt{-1}$,
$20+15\sqrt{-1}$,

124. The following principles, which
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magnitude, is impossible. The larger x become
more nearly does the value approach to zero; and
by the continual increase of x, it may be made
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finite: and, therefore, we reasonably regard $\frac{a}{0}$,
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signify a quantity that is infinitely great.

28. The expression $\frac{0}{0}$ is another remarkable
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that the quotient shall exceed 10000. Here $a=2$ and $b=10000$, and we shall have $d=\frac{2}{10000+c}$; where c may be 1, 2, 10, 100, $\frac{1}{2}$, or any other positive number. As another example, let it be required to find a divisor for 12, such that the quotient may be less than $\frac{1}{10000}$. We have here $a=12$, and $b'=\frac{1}{10000}$; and therefore $d'=\frac{12}{\frac{1}{10000}-c'}$, or $d'=\frac{120000}{1-10000c'}$; where c' may be any number less than $\frac{1}{10000}$, such as 0·00001, 0·000025, &c. Thus, if $c'=0·00002$, we should get $d'=150000$, the required divisor; and dividing 12 by this, we obtain 0·00008, which is less than $\frac{1}{10000}$ or 0·0001; and, therefore, the divisor answers.

127. The views that have now been given, will assist in enabling us to interpret the meaning of the remarkable expression, $\frac{a}{0}$, where a is a constant quantity. If a be divided by quantities, each smaller than the one preceding it, any quotient after the first will be greater than the one before it; and, by the first part of § 123., a divisor may be found, so small, that the quotient shall exceed any assigned quantity, however great. Thus, if we divide a, first by 2, then by 1, and after that by 0·1, 0·01, 0·001, and 0·000001, in succession, we get the quotients, $\frac{1}{2}a$, a, $10a$, $100a$, $1000a$, and $1000000a$; and, by taking divisors smaller and smaller, we might find quotients as great as we please, exceeding, in fact, any number, however great. Thus, if we divide by the extremely small decimal fraction, which is expressed by 1 with eleven ciphers prefixed, and which would be the millionth part of the millionth part of a unit, the quotient would be a million of millions times a. It is clear, therefore, that the more nearly the divisor approaches to zero, the greater is the quotient; and hence, $\frac{a}{0}$ is regarded as expressing a quantity greater than any thing that can be assigned, or, in other words, an *infinite quantity*.

To illustrate this subject still farther, let $\frac{a}{0}=x$: then, by multiplying by 0, and dividing by x, we get successively $a=x\times 0$, and $\frac{a}{x}=0$. Now, to make the first member of this equation absolutely equal to 0, the dividend, a, being a quantity of a fixed magnitude, is impossible. The larger x becomes, however, the more nearly does the value approach to zero; and from this limit, by the continual increase of x, it may be made to differ by the minutest quantity,—by a quantity less than any thing that can be assigned; as is plain from the second part of § 126. Hence $\frac{a}{x}$ could become zero in no other way than by x becoming infinite: and, therefore, we reasonably regard $\frac{a}{0}$, which x was assumed to represent, as being infinite. The symbol ∞ is used to signify a quantity that is infinitely great.

128. The expression $\frac{0}{0}$ is another remarkable one, which not unfrequently occurs in investigations, and which is commonly called a *vanishing fraction.*[*] Now, with regard to such an expression, if we know nothing more than what meets the eye, we can assign to it no definite value whatever: it may have, in fact, any value we please. To show this, assume it equal to x; then, by multiplying by the denominator 0, we get $0=x\times 0$, an expression which is evidently true, whatever finite value is given to x. Thus, whether x be 0, 1, 1000, -100, or any other number, positive or negative, $x\times 0=0$.

The case, however, is generally different, if we know the origin of the fraction: and it is easy to show that, for the most part, the value of such a fraction may, according to circumstances, be nothing or infinite, or a determinate quantity different from both. Thus, the fraction $\frac{(x-a)^2}{ax-a^2}$ becomes $\frac{0}{0}$, when $x=a$. By dividing its terms, however, by $x-a$, we get $\frac{x-a}{a}$, which (§ 68.) is equivalent to it. Now, when $x=a$, the numerator will vanish, while the denominator is a: the value, therefore, is zero.

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[*] This name is inappropriate, and is calculated to mislead; as it is not the *fraction*, that is evanescent, but *its terms*. Such expressions might be properly called *fractions with vanishing terms*.

The management of such fractions is, in general, greatly facilitated by means of the differential calculus. See the author's work on the subject, Section vi.

Again, the expression $\frac{x^3-a^3}{(x-a)^3}$ is also a vanishing fraction, when $x=a$; but, by dividing the numerator and denominator by $x-a$, we get $\frac{x^2+ax+a^2}{(x-a)^2}$; which becomes $\frac{3a^2}{0}$, when $x=a$; an expression which (§ 124.) is infinite. Lastly, when $x=a$, the fraction, $\frac{x^3-a^3}{x^2-a^2}$, also becomes $\frac{0}{0}$. By dividing the terms, however, by $x-a$, we get $\frac{x^2+ax+a^2}{x+a}$; which, when $x=a$, becomes the determinate quantity, $\frac{3}{2}a$.

129. From what we have just seen, it will appear, that, when a fraction, for a particular value of some quantity contained in it, becomes a vanishing one, we ought to reduce it to its lowest terms, and in the result to give that quantity the same particular value. If the fraction cannot be reduced to lower terms, its value is indeterminate.

It is plain also, that any fraction whatever may be made to assume the vanishing form, by multiplying its terms by a quantity which, on a particular supposition, would become nothing. Thus, suppose the numerator and denominator to be x^2+a^2 and $x+a$: then, if each were multiplied by $x+2a$, a fraction would be obtained, which would be a vanishing one, when $x=-2a$.

Exercises.

16. Find the value of the fraction, $\frac{2x^2+3ax-2a^2}{6x^2-ax-a^2}$, when $x=\frac{1}{2}a$. *Ans.* 1.

17. What is the value of $\frac{x^2-2x-3}{x^2-x-6}$, when $x=3$? *Ans.* $\frac{4}{5}$.

18. What does $\frac{12x^3-25x^2+9}{16x^3-13x+3}$ become, when $x=\frac{3}{4}$? *Ans.* $-\frac{69}{56}$.

19. Required the value of $\frac{x^5+a^5}{x^3+a^3}$, when $x=-a$. *Ans.* $\frac{5}{3}a^2$.

20. Find the value of $\frac{4x^3-5x^2+1}{4x^3-5x+1}$, when $x=1$. *Ans.* $\frac{2}{7}$.

21. When does $\frac{8x^3-27a^3}{4x^2-9a^2}$ become a vanishing fraction, and what is its value then?* *Ans.* It is a vanishing fraction when $x=\frac{3}{2}a$, and its value is $\frac{9}{2}a$.

22. Find the value of $\frac{x^2+x}{x^3+x^2}$, when $x=0$. *Ans.* ∞.

CHAPTER VII.

ARITHMETICAL AND GEOMETRICAL PROGRESSIONS.

130. If a series of three or more quantities be such, that, after the first, each is derived from the one immediately preceding it, by the addition of a constant quantity, these quantities are said to be in *arithmetical progression*, or to be *equidifferent;* and the constant quantity is called the *common difference* of the series. If the common difference be positive, the series is said to be *ascending;* if negative, *descending*. Thus, 1, 2, 3, 4, 5, &c., is an ascending series, having the common difference 1; and 40, 37, 34, 31, &c., is a descending one, having the common difference -3.

In series of every kind, the first and last terms are called the *extremes*, and the rest the *means*.

In series of this kind, there are five quantities which require consideration: the *two extremes*, the *common difference*, the *number of terms*, and their *sum*. Now, if we put n to denote the number of terms, we may conveniently express the terms of such a series by $t_1, t_2, t_3, \ldots\ldots t_{n-1}, t_n$; in which notation t_1 and t_n will be the extremes: and, to complete the notation, we may represent the common difference $(t_2-t_1, t_3-t_2, \ldots, t_n-t_{n-1})$ by d, and the sum of n terms by s_n.

131. This notation being adopted, we have evidently, from the definition given above, the following as the successive terms of any such series:

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* To solve this question, put the denominator equal to nothing. Then, by resolving the equation so obtained, we get $x=\frac{3}{2}a$, a value which, when substituted for x in the numerator, makes it vanish also. It is plain, that instead of putting the denominator equal to nothing, we might have put the numerator equal to it, and might have substituted the resulting value of x in the denominator.

$$t_1,\ t_1+d,\ t_1+2d,\ t_1+3d,\ \ldots\ldots,\ t_1+(n-1)d;$$

one d being added for finding each term, after the first, to the preceding term; so that, to find the nth term, $n-1$ times d are added to the first term. Hence we have $t_n=t_1+(n-1)d$.

This formula contains the four quantities, t_1, t_n, n, and d; and it will give any one of the four, when the other three are known; or, as it may be expressed, it will give any one of them *in terms of the other three.* Thus, without change, it gives the value of t_n; while by mere transposition it gives $t_1=t_n-(n-1)d$. Also, by resolving the equation for d, we get $d=\dfrac{t_n-t_1}{n-1}$; and, by similar means, we should find $n=\dfrac{t_n-t_1}{d}+1$. What has now been established will be illustrated by the following examples.

Exam. 1. Required the fifteenth term of the series 1, 3, 5, 7, &c.

Here we have $t_1=1$, $d=2$, and $n=15$; and therefore, by the last §, we have $t_{15}=1+(15-1)2=29$.

Exam. 2. Find the twenty-fifth term of the series, 100, 97, 94, 91, &c.

In this example, we have $t_1=100$, $d=-3$, and $n=25$; and, therefore, $t_{25}=100-(25-1)3=28$.

Exam. 3. In a series of forty terms, given the extremes equal to 10 and 100; to find the common difference.

Here, from the expression above obtained for d, we get

$$d=\frac{100-10}{40-1}=2\tfrac{4}{13}.$$

132. The last or nth term of the series being $t_1+(n-1)d$, and each term being less by d, than the succeeding one, it is plain, that the last but one will be $t_1+(n-2)d$; the last but two $t_1+(n-3)d$; &c. Now, by taking the sum of the first term and the last, that of the second and the last but one, that of the third and the last but two, &c., we get, in every instance, $2t_1+(n-1)d$. Hence, it follows, that *the sum of the extremes is equal to the sum of any two terms equally distant from them.*

If n, the number of terms be odd, the middle term of the series will be $t_1+\frac{1}{2}(n-1)d$, the common difference being added only half as often to find that term as to find the nth term; and, as this is exactly the half of $2t_1+(n-1)d$, the sum of the extremes found above, it follows, that, *when the number of terms is odd, the sum of the extremes is double of the middle term.*

Thus, in the series, 2, 5, 8, 11, 14, $2+14=5+11=8\times 2$.

133. We are now prepared to find the sum of an equidifferent series. Denoting, as already mentioned, the sum of n terms by s_n, and writing the series both in direct and in reversed order, we have

$$s_n=t_1+t_2+t_3+\ldots+t_{n-2}+t_{n-1}+t_n,$$

$$\text{and } s_n=t_n+t_{n-1}+t_{n-2}+\ldots+t_3+t_2+t_1.$$

Now, by the last §, the sum of each quantity in the second member of the first line, and of the term below it in the next line, is $2t_1+(n-1)d$. By the actual addition, therefore, of the two lines we get $2s_n$ equal to $2t_1+(n-1)d$ taken n times; and, consequently, by multiplying by n and dividing by 2,

$$s_n=\tfrac{1}{2}n\{2t_1+(n-1)d\}=nt_1+\tfrac{1}{2}n(n-1)d.$$

From the first form of this sum, it appears that *the sum of the series is equal to the sum of the extremes multiplied by half the number of terms.*

This equation, like the one in § 131., contains four quantities, t_1, d, n, and s_n; and therefore it will give any one of them in terms of the other three.

Exam. 4. Required the sum of 200 terms of the series 1, 4, 7, 10, &c.

Here $t_1=1$, $d=3$, and $n=200$; and, therefore, $s_{200}=200\times 1+\frac{1}{2}\times 200\times(200-1)\times 3$; or, by performing the work, $s_{200}=59900$.

Exam. 5. Find the sum of 150 terms of the series, 333, 331, 329, &c.

Here $t_1=333$, $d=-2$, and $n=150$; and, consequently, $s_{150}=\frac{1}{2}\times 150\{666+(150-1)\times -2\}=27600$.

Exam. 6. Required the sum of n terms of the series, 1, 3, 5, 7, &c.

In this question we have $t_1=1$, and $d=2$; wherefore $s_n=n\times 1+\frac{1}{2}n(n-1)\times 2=n+n^2-n=n^2$. We thus arrive at the interesting conclusion, that, in the series of odd numbers, 1, 3, 5, 7, &c., the sum of any number of terms is the square of that number. Thus, the sum of the first 10 terms is 10^2 or 100; and that of the first 100 terms, 100^2 or 10000.

Exercises.

1. Find the 10th term, and the 150th, of the equidifferent series, 3, 5, 7, 9, &c. *Ans.* 21 and 301.

2. Find the 47th and 60th terms of the series, 100, 98, 96, &c. *Ans.* 8 and -18.

3. If the extremes of an equidifferent series be 4 and 85, and the common difference 3, what is the number of its terms? *Ans.* 28.

4. What will be the common difference by which 35 equidifferent means may be inserted between the extremes 1 and 100? *Ans.* $2\frac{3}{4}$.

5. Find the sum of the first 1000 numbers, 1, 2, 3, &c.; and also the sum of the first 1000 odd numbers. *Ans.* 500,500 and 1,000,000.

6. Required the sum of 101 terms of the series, 50, 49, 48, &c. *Ans.* 0, the positive and negative terms destroying each other.

7. Find the sum of n terms of the series 3, 5, 7, 9, &c. *Ans.* n^2+2n.

8. The famous porcelain tower of Nankin is ascended by 884 steps. Now, suppose a person to ascend one of those steps, and then to descend to the ground; then to ascend two, and to descend to the ground again; then three; and so on, till he should reach the top and again descend to the ground: through how many steps would he have ascended in all? And, if each step were 6 inches in height, what would be the entire height ascended? Also, if each step were 14 inches in breadth, through what space, in a horizontal direction, would he have gone in both his ascents and descents? *Ans.* 391,170 steps; and 37 miles, 225 feet, in ascent, and the same in descent. Also the horizontal space described in both is 172 miles, 4570 feet.

9. If a person commence trade with 1000*l.*; and, if he add to this capital 100*l.* the first year, 200*l.* the second, 300*l.* the third; and so on during the first 20 years: and if he then increase it in the first of the ensuing 20 years by 200*l.*, in the second by 400*l.*, in the third by 600*l.*, and so on: how much will he be worth at the end of the whole 40 years? *Ans.* 64,000*l.*

134. When three or more quantities are such, that, after the first, each is obtained by multiplying the one immediately preceding it by the same number, those quantities are said to be in *geometrical progression*, or to be *continual proportionals;* and the multiplier is called the *common ratio*, or simply the *ratio* of the

series. If the ratio, without regard to its sign, be greater than unity, the series is said to be *ascending;* but if less, it is *descending*. Thus, the series 1, 2, 4, 8, 16, 32, &c., is an ascending geometrical one; as is also 1, -3, 9, -27, 81, &c.: the former having 2 as ratio, and the latter -3. The two following are descending series, the first having $\frac{1}{3}$ as ratio, and the second $-\frac{1}{2}$: 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, &c.; and 1, $-\frac{1}{2}$, $\frac{1}{4}$, $-\frac{1}{8}$, $\frac{1}{16}$, &c.

When the series consists of three terms, the second is said to be a *mean proportional* between the other two.

135. In series of this kind, there are five quantities chiefly to be considered, the two extremes t_1 and t_n; the number of terms, n; the sum of the series, s_n; and the common ratio or multiplier r. Now, by the definition given above, the second term will be found by multiplying the first by r, the third by multiplying the second by the same, and so on; and therefore the successive terms will be

$$t_1,\ rt_1,\ r^2t_1,\ r^3t_1, \ldots\ldots,\ r^{n-1}t_1.$$

The term before the last, found by dividing that one by r, will be $r^{n-2}t_1$; the next preceding term $r^{n-3}t_1$, &c.

We have, therefore, $t_n=r^{n-1}t_1$. This formula contains the four quantities t_1, t_n, n, and r; and, by means of it, any one of them may be found, if the other three be given. The value of n, however, cannot be obtained by means of the principles thus far established, as the finding of it requires the use of logarithms.

136. By taking the product of the first term t_1, and the last $r^{n-1}t_1$, we get $r^{n-1}t_1^2$: also, by taking the product of the second term rt_1, and the last but one $r^{n-2}t_1$, we obtain the same result: and, by multiplying the third term into the last but two, we get the same still; and so on. Hence, therefore, *the product of the extremes is equal to the product of any two terms equally distant from them.* If the number of terms be odd, the middle term of the series will be $r^{\frac{1}{2}(n-1)}t_1$, the multiplication by r being repeated only half as often to find that term, as to find the last: and, as the square of this is $r^{n-1}t_1^2$, the same as the product of the extremes found above, it follows, that, *when the number of terms is odd, the product of the extremes is equal to the square of the middle term.* Hence, if the extremes be given, the middle term is found by taking their product, and extracting its square root.

Exam. 1. Find the 7th term of the series 2, 8, 32, &c.

Here $t_1=2$, $r=4$, and $n=7$: therefore

$$t_7=4^6\times2=4096\times2=8192.$$

Exam. 2. Required the 6th term of the series $\frac{1}{3}$, $\frac{2}{9}$, $\frac{4}{27}$, &c.

We have here $t_1=\frac{1}{3}$; $r=\frac{2}{3}$, found by dividing the second term by the first; and $n=6$. Then, $n-1$ being 5, we raise $\frac{2}{3}$ to the 5th power, which we find to be $\frac{32}{243}$; and, by multiplying this by $t_1=\frac{1}{3}$, we get $\frac{32}{729}$, the required term.

Exam. 3. Find a mean proportional between 20 and 45.

The product of these is 900; the square root of which is 30, the required mean.

Exam. 4. If the extremes of a series of four terms be 27 and 64, what are the means?

Here $r^{n-1}=r^3$, is found by dividing t_n by t_1. Hence we have $r^3=\frac{64}{27}$; and, therefore, by extracting the cube root, we find r to be $\frac{4}{3}$. Lastly, by multiplying 27 by this, we get 36, and by multiplying 36 by the same, we get 48: 36 and 48, therefore, are the required means.

Hence we see, that to solve the celebrated problem in which it is required *to find two mean proportionals between two given numbers or magnitudes*, we have merely to divide the second by the first, and to extract the cube root of the quotient, which will be the ratio; and we then multiply the first extreme by this ratio to find the first mean, and this mean by the same to find the second. The equivalent geometrical operation, however, cannot be effected by means of elementary geometry.

137. For investigating the method of summing a series of this kind, we have, as in § 133.,

$$s_n=t_1+rt_1+r^2t_1+r^3t_1+\ .\ .\ .\ +r^{n-3}t_1+r^{n-2}t_1+r^{n-1}t_1.$$

Below this write its product by r, setting the second members so that like terms may stand in the same column. By this means we shall have the two lines standing thus:

$$\begin{aligned} s_n&=t_1+rt_1+r^2t_1+r^3t_1+\ .\ .\ +r^{n-3}t_1+r^{n-2}t_1+r^{n-1}t_1,\\ rs_n&=\phantom{t_1+{}}rt_1+r^2t_1+r^3t_1+\ .\ .\ +r^{n-3}t_1+r^{n-2}t_1+r^{n-1}t_1+r^nt_1.\end{aligned}$$

In these two equations all the terms of the second members are plainly the same, except the first of the first and the last of the second. Subtracting, therefore, the members of the first from those of the second, we get

$$rs_n-s_n=r^nt_1-t_1,\ \text{or}\ (r-1)s_n=(r^n-1)t_1;$$

$$\text{whence } s_n=\frac{(r^n-1)t_1}{r-1}\ .\ .\ .\ (a).$$

Had we subtracted the members of the second equation from those of the first, we should have found, in a similar manner, that

$$s_n=\frac{(1-r^n)t_1{}^{*}}{1-r}\ \ldots\ (b).$$

138. If r be a fraction less than unity, and n become infinite, r^n will vanish. For r may be put under the form $\frac{1}{r'}$, where r' is the reciprocal of r, and is therefore greater than unity. Hence we have $r^n=\frac{1}{r'^n}$. Now (§ 125.) the denominator of the second member will be infinite when n is infinite: and hence (by § 127. concluding part) the second member, and consequently its equal, r^n, will be equal to nothing. In this case, therefore, the numerator in (b) becomes simply t_1, and we have

$$s_\infty=\frac{t_1}{1-r}\ \ldots\ (c):$$

whence it appears, that *to find the sum of an infinite number of terms of a descending geometrical progression*, we subtract the ratio from unity, and divide the first term by the remainder.

Exam. 5. Required the sum of the first seven terms of the series, 1, 3, 9, 27, &c.

Here $t_1=1$, $r=3$, and $n=7$; and, by (a), $s_7=\frac{3^7-1}{3-1}=\frac{2187-1}{2}=1093$.

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[*] This latter expression would be obtained from the former by multiplying the numerator and denominator by -1, or, which is the same, by changing the signs of all the terms of both.

It may be remarked, that if x be taken equal to 1, and a be changed into r, in the formula at the end of § 58., and if the result be multiplied by t_1, we obtain $\frac{(1-r^n)t_1}{1-r}=t_1+rt_1+r^2t_1+\ldots\ldots+r^{n-1}t_1$; which agrees exactly with what has just been found. In fact, the method of summation employed above is virtually the same as the multiplying of the series by $1-r$, and is therefore a reversing of the process in division, by which the formula in § 58. is obtained.

If $r=1$, the last two expressions (a) and (b), become (§ 128.) vanishing fractions. In this case, however, we find, by reverting to § 132., that $s_n=t_1+t_1+t_1+\ldots\ldots+t_1=nt_1$, the sum required.

Exam. 6. Find the sum of the first six terms of the series, $\frac{1}{2}$, $\frac{3}{8}$, $\frac{9}{32}$, &c.

We have here $t_1=\frac{1}{2}$, $r=\frac{3}{4}$, and $n=6$. Then $r^n=(\frac{3}{4})^6=\frac{729}{4096}$. Taking this from 1, and multiplying the remainder by $\frac{1}{2}$, we find for the numerator of the second member of (b), $\frac{3367}{8192}$; while the denominator is $1-\frac{3}{4}$, or $\frac{1}{4}$. Dividing the former result by the latter, we get $s_6=\frac{3367}{2048}=1\frac{1319}{2048}$.

Exam. 7. Find the sum of nine terms of the series, 1, $-\frac{1}{2}$, $\frac{1}{4}$, $-\frac{1}{8}$, &c.

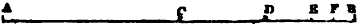
Here $t_1=1$, $r=-\frac{1}{2}$, and $n=9$. Therefore we have, by (b),

$$s_9=\frac{1-(-\frac{1}{2})^9}{1+\frac{1}{2}}=\frac{1+\frac{1}{512}}{1+\frac{1}{2}}=\frac{171}{256}.$$

Exam. 8. Required the sum of the infinite series, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c.

We have here $t_1=\frac{1}{2}$, $r=\frac{1}{2}$, and $n=\infty$. Then (§ 138.) $s_\infty=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$, the required sum.[*]

[*] As learners generally feel some difficulty in conceiving how the sum of an infinite number of terms can be a finite quantity, it may be proper to give some illustration of the subject, and the last example serves well for the purpose. Let the line AB represent the unit of which the terms of the series are the half, the fourth, the eighth, &c. Bisect AB in C, CB in D, DB in E, EB in F, &c. Then AC is one half of the given line, CD one fourth of it, DE one eighth, &c.; and, therefore, AC, CD, DE, &c., are the successive terms of the series. Now, it will be seen at once, that no finite number of these terms, however great, can amount completely to the whole line AB; as, in finding the terms by the successive bisections, there is always a portion of AB remaining unexhausted. Thus, if we take only the first term AC, there remains $CB=\frac{1}{2}$; the first and second terms amount to AD, and the unexhausted part is $DB=\frac{1}{4}$, or equal to half the last remainder CB. In like manner the sum of three terms is AE, and the unexhausted part EB, which is one half of DB, the last remainder. We thus see that there will always be a remainder, but that each remainder will be only half the preceding one. After the first term the remainder is $\frac{1}{2}$; after the second, the square of $\frac{1}{2}$; and after n terms, the nth power of $\frac{1}{2}$. Now, (§ 138.) the nth power of $\frac{1}{2}$ becomes zero, when n is infinite. Hence, the sum of an infinite number of terms of the series is equal to the unit AB, the remainder becoming evanescent. It thus appears, that *the sum of an infinite series is the amount to which the sum of its terms continually approaches, as more and more of them are taken; and from which their sum, if a sufficient number of them be employed, will differ by a quantity less than any thing that can be assigned:* and unless a



Exam. 9. Required the sum of five terms, and the sum of an infinite number of terms, of the series $\frac{9}{10}$, $\frac{9}{100}$, $\frac{9}{1000}$, &c.

number can be found which has these two distinctive properties, the series will have no finite sum, but will give an amount exceeding any number, however great, if a sufficient number of its terms be taken. Thus, in the series, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., we have seen, in the first place, that the sum approaches continually to 1, as more and more terms are taken; and, in the second place, that if a sufficient number of them be taken, their sum will differ from 1 by as small a quantity as we please: the sum, therefore, is 1. The sum of the same series also continually approaches to 2; but the difference between it and 2 cannot be made as small as we please, as it must always exceed 1; and, therefore, 2 is not the sum, as it has only one of the two properties mentioned above.

There may plainly be numberless series, in each of which the sum of an infinite number of terms will be infinitely great. Such is the series, 1, 1, 1, &c.; as also, 1, 2, 3, &c. It may be remarked, also, that a series, the terms of which go on continually diminishing, and becoming smaller and smaller without limit, has not necessarily a finite sum. Thus, as will be shown hereafter in the Chapter on Series, so many terms of the series, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., may be taken, that their sum shall exceed any number, however great.

An infinite series which has a finite sum is said to *converge*, or to be a *convergent* one.

The illustration given above is very plain and palpable. We shall arrive, however, at the same conclusion more briefly thus: by formula (*b*), we shall have in the series, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. $s_n = 1 - (\frac{1}{2})^n$. Hence we see that the sum of any number of terms, n, will always be less than 1 by the nth power of $\frac{1}{2}$; and, since (§ 138.) n may be taken so great as to render the nth power of $\frac{1}{2}$ as small as we please, the sum will tend to 1 as its limit, differing from it by a quantity less than any thing that can be assigned, when n is taken sufficiently large.

It may be remarked, that if 1 be divided by 1 under the form $2-1$, the quotient will be the series which we have been considering. In like manner, if 1 be divided by 2 under the form $3-1$, the quotient will be the infinite series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} +$, &c.; the sum of which is therefore $\frac{1}{3-1}$ or $\frac{1}{2}$. So likewise, if we divide 3 by 1 under the form, $4-3$, we get the infinite series, $\frac{3}{4} + \frac{9}{16} + \frac{27}{64} +$, &c.; the sum of which, accordingly, is 3. Again, if 1 be divided by 3 in the form $2+1$, the quotient will be $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} +$, &c.; and the sum of this is $\frac{1}{3}$. In infinite series, such as this, having the terms alternately positive and negative, the values of one term, of two terms, of three, &c., are alternately greater and less than the sum of the series; to which, however, they continually approach. Thus, in the last series, the first term is $\frac{1}{2}$: which exceeds the sum $\frac{1}{3}$ by $\frac{1}{6}$: the first and second together make $\frac{1}{4}$, which is less than $\frac{1}{3}$ by $\frac{1}{12}$; the first three terms amount to $\frac{3}{8}$, which is greater than $\frac{1}{3}$ by $\frac{1}{24}$, &c. This series, it may be farther remarked, is evidently the difference of the two series, $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} +$, &c., and $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} +$, &c.; and as the latter of these is plainly half the former, each of its terms being half the corresponding one in the other, Here we have $t_1 = \frac{9}{10}$, and $r = \frac{1}{10}$; and, therefore, $r^5 = \frac{1}{100000}$. Subtracting this from 1, we get $\frac{99999}{100000}$, which is the sum of five terms, as, according to (b), it should be multiplied by $\frac{9}{10}$ and divided by $1 - \frac{1}{10}$; and these being the same, the two operations neutralise one another. Again, for the sum of the infinite series, we divide $\frac{9}{10}$ by $1 - \frac{1}{10}$, and the quotient is 1, the sum required. We thus see that the value of the interminate decimal ·9999...., or, as it may be written,

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \text{\&c., is } 1.$$

Exercises. In each of the following exercises, find the sum of as many terms as are denoted by the number attached to s in the answer of the same exercise.

Exercises.	Answers.	Exercises.	Answers.
1. 1, 4, 16, &c.	$s_6 = 1365$.	3. $\frac{1}{27}, \frac{1}{9}, \frac{1}{3}$, &c.	$s_7 = 40\frac{13}{27}$.
2. 3, 6, 12, &c.	$s_{16} = 196605$.	4. $\frac{3}{2}, \frac{9}{4}, \frac{27}{8}$, &c.	$s_8 = 73\frac{227}{256}$.

5. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}$, &c. *Ans.* $s_8 = 1\frac{6049}{6561}$, and $s_\infty = 2$.

6. $\frac{11}{10}, (\frac{11}{10})^2, (\frac{11}{10})^3$, &c. *Ans.* $s_6 = 8{\cdot}487171$.

7. $\frac{10}{11}, (\frac{10}{11})^2, (\frac{10}{11})^3$, &c. *Ans.* $s_6 = 4\frac{629366}{1771561}$, and $s_\infty = 10$.

8. $\frac{1}{3}, -\frac{2}{9}, \frac{4}{27}, -\frac{8}{81}$, &c. *Ans.* $s_5 = \frac{55}{243}$, and $s_\infty = \frac{1}{5}$

9. 1, -2, 4, -8, &c. *Ans.* $s_n = \frac{1}{3}\{1 - 2^n \times (-1)^n\}$.

10. $\frac{1}{3}, -(\frac{1}{3})^3, +(\frac{1}{3})^5$, &c. *Ans.* $s_4 = \frac{656}{2187}$, and $s_\infty = \frac{3}{10}$.

11. If a person buy twenty horses, paying a farthing for the first, a halfpenny for the second, a penny for the third, and so on, always giving for each horse, after the first, the double of what he paid for the preceding one, how much would the whole cost? *Ans.* £1092 5*s.* 3¾*d.*

12. If there had been trebling instead of doubling in the last question, what would have been the cost of the twenty horses, the first still costing a farthing? *Ans.* £1,816,033 10*s.* 10*d.*

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the original series is the same as $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} +$, &c. It would be reduced to this also by subtracting each negative term from the positive one immediately preceding it.

CHAPTER VIII.

RESOLUTION OF SIMULTANEOUS EQUATIONS.

139. We have already seen (Chap. I. and V.) the mode of resolving a single equation of the first degree containing one unknown quantity: and we may now proceed to investigate the method of finding the values of two or more unknown quantities. We shall see in what follows, that, for limiting a question, there must always be as many equations given, as there are unknown quantities to be determined: and, as these equations coexist, so as to be all true at the same time, they have been called *simultaneous equations.*

For resolving questions in which two unknown quantities are to be found, four methods have been given by algebraists; and it will be proper to illustrate all these, as sometimes one of them is preferable, and sometimes another.

140. By *eliminating a quantity between two or more equations*, is meant the deriving of one or more other equations from them which shall not contain that quantity. This may often be effected very simply, in case of two equations, by multiplying them by such numbers or quantities, as shall give results having the terms which contain the quantity to be eliminated, the same in both, except that their signs may be the same or opposite; then, by taking the difference or sum of the results, according as those terms have the same or contrary signs, the required equation will be obtained.*

Exam. 1. Let it be required to find the values of x and y in the two equations $6x+25y=37$, and $9x-10y=8$; that is, to find such values of those quantities, as shall satisfy the two equations simultaneously.

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* This, for brevity, is often called *the method of addition and subtraction.* It is plain, that by using, when necessary, a negative multiplier, we might make the terms to be eliminated exactly the same, so that subtraction might always be employed.

Here, equation (3.) is found from (1.) by multiplying by 3, and (4.) from (2.) by multiplying by 2. Now, (3.) and (4.) having each $18x$ with the same sign, we subtract the one from the other; thus destroying that term, and consequently eliminating x, as the resulting equation $95y=95$ contains only y: and from this (§ 26.) by dividing by 95, we get $y=1$.

$$6x+25y=37 \ldots\ldots (1.)$$
$$9x-10y=8 \ldots\ldots (2.)$$
$$18x+75y=111 \ldots\ldots (3.)$$
$$18x-20y=16 \ldots\ldots (4.)$$
$$95y=95 \ldots\ldots (5.)$$
$$y=1 \ldots\ldots (6.)$$
$$6x+25=37 \ldots\ldots (7.)$$
$$x=2 \ldots\ldots (8.)$$
$$9x-10=8 \ldots\ldots (9.)$$
$$x=2 \ldots\ldots (10.)$$

Equation (7.) is obtained from (1.) by taking $y=1$; and thence, by transposition, &c. we get $x=2$. Equation (9.), again, is the same as (2.) with y taken $=1$; and hence we have another mode of finding x. The agreement of the results proves the correctness of the part of the work that follows equation (4.). To give a complete proof of the correctness of the results, we have merely to show that the values found for x and y satisfy both the given equations. Thus, since $x=2$ and $y=1$, the first member of (1.) becomes $12+25$, or, by addition 37, which is what it ought to be; and, by a like substitution, the first member of (2.) becomes $18-10$, or, by contraction 8, which is the same as the second member. This method of proof is applicable in all cases.

The principles on which the process depends are quite obvious. Thus, equations (3.) and (4.) are true by Axiom 3. p. 9., and (5.) by Axiom 2. The rest of the process all depends on what has been already shown regarding equations having one unknown quantity each. If we wished to eliminate y instead of x, the operation would be as in the margin. In this process (3.) and (4.) are obtained from (1.) and (2.) by multiplying them respectively by 2 and 5. Then, as (3.) and (4.) have the terms containing y the same, but with opposite signs, these two terms are destroyed by the addition of the two equations, and we get $57x=114$, an equation which does not contain y, and which, by division by 57, gives $x=2$, as before. We then find y by substituting the value of x in (1.), thus getting equation (7.); and it

$$6x+25y=37 \ldots\ldots (1.)$$
$$9x-10y=8 \ldots\ldots (2.)$$
$$12x+50y=74 \ldots\ldots (3.)$$
$$45x-50y=40 \ldots\ldots (4.)$$
$$57x=114 \ldots\ldots (5.)$$
$$x=2 \ldots\ldots (6.)$$
$$12+25y=37 \ldots\ldots (7.)$$
$$y=1 \ldots\ldots (8.)$$

might be found with equal facility, by substituting the same value in (2.).

141. In the practical application of this principle, we may always employ as multipliers the coefficients of the quantity to be eliminated, taking them in a reversed order. When these have a common measure, however, they should be divided by it, and, for simplicity, the quotients should be employed instead of the coefficients themselves. Thus, in the present example, for eliminating x, the coefficients of that quantity being 6 and 9, we might have multiplied (1.) by 9, and (2.) by 6, thus employing the coefficients. It is better, however, to divide these by 3, and to use the quotients, 3 and 2. In like manner, for eliminating y, the coefficients of that quantity being 25 and 10, we might multiply the first equation by 10, and the second by 25. Dividing these, however, by 5, we use in preference the quotients, 2 and 5.

Exam. 2. Resolve the two general equations, (1.) and (2.) in the margin.

Here we multiply (1.) by b_2 to find (3.), and (2.) by b_1 to find (4.). The difference of the results is (5.): and from this (6.) is obtained by dividing by $a_1b_2-a_2b_1$, the coefficient of x. We then multiply (1.) by a_2 to find (7.), and (2.) by a_1 to find (8.). The difference of (7.) and (8.) is (9.); and (10.) is obtained from this by dividing by the coefficient of y.

$$a_1x+b_1y=c_1 \ldots\ldots\ldots\ldots (1.)$$

$$a_2x+b_2y=c_2 \ldots\ldots\ldots\ldots (2.)$$

$$a_1b_2x+b_1b_2y=b_2c_1 \ldots\ldots\ldots (3.)$$

$$a_2b_1x+b_1b_2y=b_1c_2 \ldots\ldots\ldots (4.)$$

$$(a_1b_2-a_2b_1)x=b_2c_1-b_1c_2 \ldots\ldots (5.)$$

$$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1} \ldots\ldots (6.)$$

$$a_1a_2x+a_2b_1y=a_2c_1 \ldots\ldots\ldots (7.)$$

$$a_1a_2x+a_1b_2y=a_1c_2 \ldots\ldots\ldots (8.)$$

$$(a_1b_2-a_2b_1)y=a_1c_2-a_2c_1 \ldots\ldots (9.)$$

$$y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}\text{*} \ldots\ldots (10.)$$

In taking the difference of (3.) and (4.), and of (8.) and (9.), it is of no consequence which is taken from the other: as the results obtained by the two variations of the process in each case would be the same, except that the signs of the numerator and those of the denominator would be all changed into the contrary

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[*] This expression for y would be found from that of x before obtained, by interchanging a_1 and b_1, and also a_2 and b_2 throughout, and then changing the signs of the numerator and denominator.

ones, which (§ 137.) would make no difference in the values of these results.

142. We can now see that two independent equations, and only two, are required for finding two unknown quantities. Thus, in each of the foregoing examples, we employ both the given equations, and nothing more: and were there but one equation, we could not find either x or y. In Exam. 1., for instance, if only equation (1.) were given, we should find $x=\frac{37-25y}{6}$, by transposing $25y$, and dividing by 6. This expression, however, we cannot compute, as it contains y, which is unknown. With only one equation, in fact, the question is indeterminate; as there may be found an infinite number of values of x and y that will satisfy the equation. Thus, if in the expression just found for x, we take $y=0$, we get $x=6\frac{1}{6}$; and if these two values of x and y be substituted for them in equation (1.) they will satisfy it. If, again, $y=\frac{1}{2}$, we should find $x=4\frac{1}{12}$; and these values would also answer. So, likewise, if $y=-1$, we should get $x=10\frac{1}{3}$: and thus we might proceed without limit. The consideration of indeterminate equations will be resumed hereafter.

143. A *second method* * of eliminating a quantity between two equations, is to find an expression for it from each of them, and to put the two expressions equal to each other; as the result will be an equation not containing that quantity.

Exam. 3.

$$3x-2y=11 \quad \ldots\ldots (1.)$$
$$2x+3y=29 \quad \ldots\ldots (2.)$$
$$x=\frac{11+2y}{3} \quad \ldots\ldots (3.)$$
$$x=\frac{29-3y}{2} \quad \ldots\ldots (4.)$$
$$\frac{11+2y}{3}=\frac{29-3y}{2} \quad . (5.)$$
$$22+4y=87-9y \quad . . (6.)$$
$$y=5 \quad \ldots\ldots (7.)$$
$$3x-10=11 \quad . . (8.)$$
$$x=7 \quad \ldots\ldots (9.)$$

In this example, (3.) is found from (1.) by transposing $-2y$, and dividing by 3; and (4.) from (2.) by transposing $3y$, and dividing by 2. The values of x in (3.) and (4.) are then put equal to one another, and give equation (5.). By clearing this equation of fractions, we get (6.); and thence the value of y is readily found to be 5. Equation (8.) is the same as (1.), with the substitution of 5 for y; and from this we get $x=7$ in the common way. This value of x might also be found in a similar manner from (2.). Another solution would be obtained, by finding expressions for y from (1.) and (2.), and equalling them: as y would thus be eliminated.

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* This method of eliminating has been called the *method of comparison*.

144. A *third method* of elimination is, to find an expression from one of the equations for the quantity to be eliminated, and to substitute that value instead of it in the other.*

In this example, from equation (1.) we get (3.) by transposition. Then, as $-3y$ occurs in (2.), we multiply (3.) by 3, to find (4.). Substituting the value of $3y$ thus found in (2.), that is, subtracting it from $5x$, we get (5.), an equation which contains only x; and from it x is found in the usual way to be 2. Lastly, by substituting this value of x in (3.) we get equation (7.), which gives the value of y. The steps of the process are all self-evident.

Exam. 4.

$2x+y=7$ . . . . (1.)
$5x-3y=1$ . . . . (2.)
$y=7-2x$ . . . . (3.)
$3y=21-6x$ . . . . (4.)
$5x-21+6x=1$ . . . . . (5.)
$x=2$ . . . . (6.)
$y=7-4=3$ . . . . (7.)

It may be remarked also, that the problem might be solved, but not so easily, by finding the value of y from the second equation, and substituting it in the first; by finding the value of x from the first, and substituting it in the second; or, finally, by finding the value of x from the second, and substituting it in the first. The facility of the mode above adopted, arises from the circumstance, that the coefficient of y in (1.) is unity.

In this example, equation (3.), which gives an expression for x, is found from (1.), by transposing $5y$ and dividing by 7. To prepare for substitution, (3.) is multiplied by 2, the coefficient of x in (2.). Then the value of $2x$, thus found, is substituted for that quantity in (2.); and thus we get (5.), an equation which does not contain x, and from which we obtain y in the usual way. Then x is found at once from (3.) by substituting 4 for y.

Exam. 5.

$7x-5y=1$ . . . . (1.)
$2x+3y=18$ . . . . (2.)
$x=\frac{1+5y}{7}$ . . . . (3.)
$2x=\frac{2+10y}{7}$ . . . . (4.)
$\frac{2+10y}{7}+3y=18$ . . . . . (5.)
$2+10y+21y=126$ . . . . (6.)
$y=4$ . . . . (7.)
$x=\frac{1+20}{7}=3$ . . . . . . (8.)

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* This has been called the *method of substitution*.

The work might be varied by finding the value of x from (2.) and substituting the result in (1.); or by finding an expression for y from either (1.) or (2.), and substituting it in the other.

145. A *fourth method* is to multiply one of the equations by a quantity, m, and to add the members of the result to those of the other equation. Then, by putting the coefficient of y in the resulting equation equal to nothing, we shall eliminate y, and x will be found in the usual way: and, again, by putting the coefficient of x in the same equation equal to nothing, x will be eliminated, and the value of y will be obtained.

Exam. 6.

$$\begin{array}{rl}
2x+3y=16 \dots\dots\dots\dots & (1.) \\
3x-2y=11 \dots\dots\dots\dots & (2.) \\
2mx+3my=16m \dots\dots\dots & (3.) \\
(2m+3)x+(3m-2)y=16m+11 \dots\dots & (4.) \\
3m-2=0 \dots\dots\dots\dots & (5.) \\
m=\frac{2}{3} \dots\dots\dots\dots & (6.) \\
(\frac{4}{3}+3)x=\frac{32}{3}+11 \dots\dots\dots & (7.) \\
x=5 \dots\dots\dots\dots & (8.) \\
2m+3=0 \dots\dots\dots\dots & (9.) \\
m=-\frac{3}{2} \dots\dots\dots\dots & (10.) \\
(-\frac{9}{2}-2)y=-24+11 \dots\dots & (11.) \\
y=2 \dots\dots\dots\dots & (12.)
\end{array}$$

In this example (3.) is obtained from (1.) by multiplying by m: and the equation so found will obviously be true, whatever may be the value of m; so that we may give m any value that may answer the object in view. Then equation (4.) is found from (2.) and (3.) by addition. Now, it is plain, that the second term of this will vanish, if $3m-2=0$; and the first, if $2m+3=0$: and these are equations (5.) and (9.). By resolving the first of these we get $m=\frac{2}{3}$; which is the value of m, out of all the infinite number of values which it may have, that will destroy the second term of equation (4.). Taking m, therefore, equal to $\frac{2}{3}$ in (4.), wanting the second term, we get (7.), an equation which does not contain y, and from which x is found in the usual way. From equation (9.), again, we get $m=-\frac{3}{2}$; and by substituting this for m in (4.), and omitting the first term, we obtain (11.), which gives $y=2$.

146. When fractions or radicals occur, they must be removed by the methods pointed out in §§ 110, &c., and the results must be simplified, if possible, by transposition or any other means that may be admissible, before the principles that have just been

given can be employed. In a few instances, which will be best learned by practice, such as in Exercise 10., the removal of fractions and radicals may be advantageously dispensed with, but in general they must be removed.

It may be remarked, also, that the values for x and y, obtained in Exam. 2., will serve for solving all questions of the kind which we have been considering, when the equations have been reduced to the proper form. Thus, in Exam. 1. we have $a_1=6$, $b_1=25$, $c_1=37$; $a_2=9$, $b_2=-10$, and $c_2=8$; and, therefore,

$$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}=\frac{-10\times37-8\times25}{6\times-10-9\times25}=2, \text{ and}$$

$$y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}=\frac{6\times8-9\times37}{6\times-10-9\times25}=1;$$

the same as the values found by the particular solution of the example.

*Exercises.**

Find the values of x and y in the following simultaneous equations.

Exercises.	Answers.	Exercises.	Answers.
1. $x+3y=17,$ $3x+y=11.$	$x=2,$ $y=5.$	6. $\frac{x+2y}{3}+x=22,$ $\frac{3x-2y}{5}+y=12.$	$x=13,$ $y=7.$
2. $x+3y=13,$ $3x-y=9.$	$x=4,$ $y=3.$	7. $\frac{2x-3y}{2}+x=2,$ $\frac{6x+y}{3}-x=1.$	$x=1,$ $y=0.$
3. $x+\frac{1}{2}y=16,$ $\frac{1}{2}x-\frac{1}{3}y=1.$	$x=10,$ $y=12.$	8. $\frac{x-1}{2}-\frac{y+1}{3}=1,$ $\frac{x+2}{3}+\frac{y-1}{4}=4.$	$x=7,$ $y=5.$
4. $\frac{3}{4}x-\frac{1}{3}y=1,$ $\frac{2}{3}x+\frac{3}{4}y=26.$	$x=12,$ $y=24.$		
5. $x+y=27,$ $\frac{2x-y}{3}-\frac{1}{2}y=6.$	$x=19,$ $y=8.$		

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* The student should perform these exercises, as well as those that follow them, in which the values of three unknown quantities are to be found, partly by one of the foregoing methods, and partly by the others, practising in the use of each method, till he feels himself sufficiently acquainted with the mode of employing it. He will find it useful, also, to work a number of the exercises by all the rules, as he will thus see the relations which the rules bear to one another; and he ought to perform several of them by means of the formulas obtained in Exam. 2.

Exercises.	Answers.	Exercises.	Answers.
9. $\left.\begin{aligned}&3x+5y=68,\\&\frac{x+y}{9}=\frac{x-y}{2}.\end{aligned}\right\}$	$x=11,$ $y=7.$	11. $\left.\begin{aligned}&\frac{x+2y}{3}+2x=10,\\&\frac{3x+2y}{2}=19-4y.\end{aligned}\right\}$	$x=3\frac{1}{2},$ $y=2\frac{3}{4}.$
10. $\left.\begin{aligned}&\frac{1}{x}+\frac{1}{y}=5,\\&\frac{5}{x}-\frac{3}{y}=1.\end{aligned}\right\}$ *	$x=\frac{1}{2},$ $y=\frac{1}{3}.$	12. $\left.\begin{aligned}&\frac{x+1}{3}=\frac{y-1}{2},\\&\frac{x+y}{2}=\frac{x-y}{4}+5.\end{aligned}\right\}$	$x=5,$ $y=5.$

Exercises.	Answers.
13. $\left.\begin{aligned}&x+\tfrac{1}{3}y=y-2,\\&y+\tfrac{1}{4}x=x+6.\end{aligned}\right\}$	$x=4,$ $y=9.$
14. $\left.\begin{aligned}&\frac{x+y}{2}-\frac{2x-y}{5}=4,\\&\frac{5x-y}{4}+3=\frac{4x+y}{2}-3.\end{aligned}\right\}$	$x=2\frac{2}{3},$ $y=5\frac{1}{3}.$
15. $\left.\begin{aligned}&x+ay=b,\\&ax+y=c.\end{aligned}\right\}$	$x=\frac{ac-b}{a^2-1},\quad y=\frac{ab-c}{a^2-1}.$
16. $\left.\begin{aligned}&\frac{100-x}{3}=\frac{100+y}{4},\\&\frac{50-x}{4}=\frac{50+y}{7}.\end{aligned}\right\}$	$x=10,$ $y=20.$
17. $\left.\begin{aligned}&\sqrt{(x^2+2y-1)}-1=x,\\&\sqrt{(y^2+3x)}-1=y.\end{aligned}\right\}$	$x=3,$ $y=4.$
18. $\left.\begin{aligned}&ax+by=c,\\&(a+a')x+(b+b')y=c+c'.\end{aligned}\right\}$	$x=\frac{b'c-bc'}{ab'-a'b},$ $y=\frac{ac'-a'c}{ab'-a'b}.$
19. $\left.\begin{aligned}&ax+\frac{y}{b}=c,\\&a'x+\frac{y}{b'}=c'.\end{aligned}\right\}$	$x=\frac{bc-b'c'}{ab-a'b'},$ $y=\frac{abb'c'-a'bb'c}{ab-a'b'}.$

147. *When more than two unknown quantities are concerned,* the elimination can be effected by means of the same general

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* Or $x^{-1}+y^{-1}=5$, and $5x^{-1}-3y^{-1}=1$. The solution will be obtained most easily by finding x^{-1} and y^{-1}, and taking their reciprocals.

principles that have been explained with regard to two equations containing two unknown quantities. This will be illustrated in the following example.

Here, to eliminate x between the given equations, (1.), (2.), and (3.), we first eliminate it between (1.) and (2.), multiplying the first by 3 and the second by 2. We thus obtain (4.) and (5.); and, by taking the former of these from the latter, we get (6.), an equation containing only y and z. We next eliminate x between (1.) and (3.), by multiplying the former by 2, and subtracting the result (7.) from the latter, and thus obtaining (8.). We have now, therefore, two equations (6.) and (8.), containing two unknown quantities, y and z; and the rest of the work may be carried out by means of any of the four methods established above. As it stands here, (9.) is found from (8.) by multiplying by 13; and (10.) from (6.) and (9.) by subtracting: while (12.) is found from (8.) and (11.); and (14.) from (1.), (13.), and (11.).

Exam. 7.

$$2x - 3y + 4z = 3 \ldots\ldots (1.)$$
$$3x + 2y - 5z = 13 \ldots\ldots (2.)$$
$$4x - 5y + 3z = 4 \ldots\ldots (3.)$$
$$6x - 9y + 12z = 9 \ldots\ldots (4.)$$
$$6x + 4y - 10z = 26 \ldots\ldots (5.)$$
$$13y - 22z = 17 \ldots\ldots (6.)$$
$$4x - 6y + 8z = 6 \ldots\ldots (7.)$$
$$y - 5z = -2 \ldots\ldots (8.)$$
$$13y - 65z = -26 \ldots (9.)$$
$$43z = 43 \ldots\ldots (10.)$$
$$z = 1 \ldots\ldots (11.)$$
$$y - 5 = -2 \ldots (12.)$$
$$y = 3 \ldots\ldots (13.)$$
$$2x - 9 + 4 = 3 \ldots\ldots (14.)$$
$$x = 4 \ldots\ldots (15.)$$

The work, according to this method, would admit of several variations. Thus, x might be eliminated between (1.) and (2.), and between (2.) and (3.); or between (1.) and (3.), and between (2.) and (3.). We might also commence by eliminating either y or z instead of x; and the effecting of each of these eliminations, like the foregoing elimination of x, would admit of three variations.

The solution might also be effected by means of any of the three remaining methods, with variations corresponding to those just mentioned. Thus, suppose we wished to eliminate z by the second method, we should find expressions for that quantity from each of the three given equations. Then, by setting the first of these equal to the second, and also the first equal to the third, we should get two equations containing only x and y: and one variation would be obtained by equalling the first and second values of z, and also the second and third; and another by equalling the first and third and the second and third. In case of variations in this and other instances, the student, after some practice, will frequently be able to see beforehand which of the methods will give the simplest solution. On this point, it may be stated as a general principle, to which there are few exceptions, that the simplest expressions, such as those containing the smallest numbers, should be used as much as possible, in preference to ones of a more complicated kind.

Suppose, again, that we wish to eliminate y by the third method, we have only to find its value in any one of the three given equations, and to substitute that value for it in the other two; as the results will be two equations, containing only x and z.

In working by the fourth method, we may multiply (1.) by m and (2.) by n, and add together the results and (3.). In the sum we can eliminate any two of the quantities, x, y, and z, by putting their coefficients equal to nothing, thus destroying two terms of the equation. Then, in what remains undestroyed, we substitute the values found for m and n in the equations obtained by putting, as above mentioned, the coefficients equal to nothing; and from the result we get at once the value of the uneliminated quantity.

Exercises. Find the values of x, y, and z, in the following sets of simultaneous equations.

Exercises.	Answers.	Exercises.	Answers.
20.		22.	
$\left.\begin{array}{l} x+2y+3z=34, \\ 2x-3y+4z=19, \\ 3x+4y-5z=-6. \end{array}\right\}$	$x=3$, $y=5$, $z=7$.	$\left.\begin{array}{l} x+y+z=6, \\ 3x-y+2z=7, \\ 4x+3y-z=7. \end{array}\right\}$	$x=1$, $y=2$, $z=3$,
21.		23.	
$\left.\begin{array}{l} 5x-4y+2z=48, \\ 3x+3y-4z=24, \\ 2x-5y+3z=19. \end{array}\right\}$	$x=10$, $y=2$, $z=3$.	$\left.\begin{array}{l} x+\frac{1}{2}y=10-\frac{1}{3}z, \\ \frac{1}{2}(x+z)=9-y, \\ \frac{1}{4}(x-z)=2y-7. \end{array}\right\}$	$x=7$, $y=4$, $z=3$.

148. The general modes of effecting elimination in the simple and more elementary cases have now been given. In many particular instances, however, solutions may be obtained more easily and elegantly by other means. This is particularly the case when the equations resemble one another in form and structure, and especially when more than two unknown quantities are concerned. The following examples and exercises will exhibit instances of this kind.

In this operation we derive (3.) from the given equations (1.) and (2.), by subtraction; and (4.) is derived from (1.) and (3.) by the same means. Lastly, (5.) is derived from (3.) by transposition, &c.

Exam. 8.

$x+2y=7$ . . . . (1.)
$2x+3y=12$ . . . . . (2.)
$x+y=5$ . . . . (3.)
$y=2$ . . . . (4.)
$x=5-y=3$ . . . . (5.)

Here, from (1.) and (2.), we get (3.), according to § 57., by division. Then the two expressions marked (4.) are obtained from (2.) and (3.) by § 52.; and, by squaring the results thus found, we get x and y.

Exam. 9.

$x-y=16$ . . . . . (1.)
$\sqrt{x}-\sqrt{y}=2$ . . . . (2.)
$\sqrt{x}+\sqrt{y}=8$ . . . . (3.)
$\sqrt{x}=5$, and $\sqrt{y}=3$. (4.)
$x=25$, and $y=9$ . . (5.)

In Exam. 10. we find (4.) by adding together equations (1.), (2.), and (3.), and taking half the result, putting s for brevity, here and in some other instances, to denote half the sum of a, b, and c. Then (5.) is found from (3.) and (4.) by subtraction: and, in the same manner, (6.) is obtained from (2.) and (4.), and (7.) from (1.) and (4.). Hence, to get x, y, and z, we are to find half the sum of a, b, and c, and from it to subtract, successively, c, b, and a.

Exam. 10.

$x+y=a$ . . (1.)
$x+z=b$ . . (2.)
$y+z=c$ . . (3.)
$x+y+z=s$ (4.)
$x=s-c$ . . (5.)
$y=s-b$ . . (6.)
$z=s-a$ . . (7.)

In Exam. 11. we get (4.) from the three given equations, where, as in the last example, $a+b+c$ is denoted by $2s$. Then the values of x, y, and z are obtained by subtracting (1.), (2.), (3.) severally from (4.), and halving each of the remainders.

Exam. 11.

$-x+y+z=a$ . . . (1.)
$x-y+z=b$ . . . (2.)
$x+y-z=c$ . . . (3.)
$x+y+z=2s$ . . (4.)
$x=s-\frac{1}{2}a$ (5.)
$y=s-\frac{1}{2}b$ (6.)
$z=s-\frac{1}{2}c$ (7.)

Here (4.) is found by taking the continual products of the members of (1.), (2.), and (3.), and extracting the square roots of the results. Then (5.) is obtained by dividing (4.) by (3.); and y would be found similarly from (4.) and (2.), and z from (4.) and (1.).

Exam. 12.

$xy=a$ . . . . (1.)
$xz=b$ . . . . (2.)
$yz=c$ . . . . (3.)
$xyz=a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$ . . . . (4.)
$x=a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-\frac{1}{2}}=\sqrt{\frac{ab}{c}}$ . . . (5.)

Exercises. Resolve the following sets of equations, without employing any of the four general methods that have been given.

24. $$\left.\begin{array}{l} x+y=a, \\ x^2-y^2=b. \end{array}\right\} \qquad \textit{Ans. } x=\tfrac{1}{2}\left(a+\frac{b}{a}\right), \quad y=\tfrac{1}{2}\left(a-\frac{b}{a}\right).$$

25. $$\left.\begin{array}{l} 3x-4y=5, \\ 5x-8y=7. \end{array}\right\} \qquad \textit{Ans. } x=3, \quad y=1.$$

26. $$\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}+\frac{1}{z}=b, \\ \frac{1}{y}+\frac{1}{z}=c. \end{array}\right\} \qquad \textit{Ans. } x=\frac{1}{s-c}, \quad y=\frac{1}{s-b}, \quad z=\frac{1}{s-a}.$$

27. $$\left.\begin{array}{l} x+y+z=9, \\ x+2y+3z=20, \\ x+3y+6z=35. \end{array}\right\} \qquad \textit{Ans. } x=2, \quad y=3, \quad z=4.$$

28. $$\left.\begin{array}{l} x^2yz=a, \\ xy^2z=b, \\ xyz^2=c. \end{array}\right\} \qquad \textit{Ans. } x=\frac{a}{a^{\frac{1}{4}}b^{\frac{1}{4}}c^{\frac{1}{4}}}, \quad y=\frac{b}{a^{\frac{1}{4}}b^{\frac{1}{4}}c^{\frac{1}{4}}}, \quad z=\frac{c}{a^{\frac{1}{4}}b^{\frac{1}{4}}c^{\frac{1}{4}}}.$$

29. $$\left.\begin{array}{r} -x+y+z+v=a, \\ x-y+z+v=b, \\ x+y-z+v=c, \\ x+y+z-v=d. \end{array}\right\} \qquad \textit{Ans. } x=\tfrac{1}{2}(s-a),^{*} \quad y=\tfrac{1}{2}(s-b), \quad z=\tfrac{1}{2}(s-c), \quad v=\tfrac{1}{2}(s-d).$$

149. This Chapter may be terminated by some miscellaneous matter, and particularly by some examples and exercises, illustrative of the principles that have been established in this and some of the preceding Chapters.

Exam. 13. Let it be required to resolve the two equations,

$$6x+9y=27, \text{ and } 8x+12y=36,$$

by the formulas found in Exam. 2.

By substituting for a_1, b_1, &c., 6, 9, &c., we get

$$x=\frac{12\times27-9\times36}{6\times12-8\times9}=\frac{0}{0}, \text{ and } y=\frac{6\times36-8\times27}{6\times12-8\times9}=\frac{0}{0}.$$

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* Here $s=\frac{1}{3}(a+b+c+d)$.

Now, the values of x and y in Exam. 2., being fractions in their lowest terms, and the results obtained in the present question being vanishing fractions, it follows, from § 126., that the question is indeterminate. In fact, the two equations are not independent of one another, the second being formed from the first by increasing each of its terms by one third of itself; so that (§ 142.) the question is indeterminate.[*]

Exam. 14. Let it be required to resolve the equations

$$12x+20y=9, \text{ and } 15x+25y=11,$$

by means of equations (6.) and (10.) page 117.

Here, by the proper substitutions, we get

$$x=\tfrac{5}{0}=\infty, \text{ and } y=-\tfrac{3}{0}=-\infty.$$

Hence it appears, that there are no finite values for x and y, that will satisfy the two equations. The equations, in fact, are incompatible with one another; and if they had their origin from any question, that question would be absurd. That they are incompatible with one another, will appear evident from the circumstance, that the first member of the first equation is four fifths of the first member of the second, while there is not the same relation between the second members.[†]

Exam. 15. The following instructive problem has often been resolved and discussed. Two couriers, A and B, start at the

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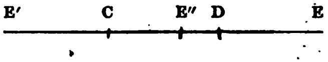
[*] By putting the denominator and numerator of (6.) in Exam. 2. each $=0$, we should find, that

$$\frac{a_2}{a_1}=\frac{b_2}{b_1}, \text{ and } \frac{b_2}{b_1}=\frac{c_2}{c_1}; \text{ and, consequently, } \frac{a_2}{a_1}=\frac{b_2}{b_1}=\frac{c_2}{c_1};$$

and the same conclusions would be obtained from (10.) in the same example. If now we put each of these three equal quantities equal to q, it will appear that, universally, the case in which the values of x and y will be vanishing fractions, will be that in which the numbers a_2, b_2, c_2 in the second equation are derivable from those in the first by multiplying each of them by $q(=\frac{4}{3}$ in the example given above).

[†] This might be readily shown in a general way, by a process nearly the same as the one employed in the preceding note. That the values of x and y satisfy the given equations in an algebraical sense, will be seen by substituting them in those equations. By this means, we get in the first, $\frac{12\times5}{0}-\frac{20\times3}{0}=9$: whence, by multiplying by the denominator, 0, we obtain $60-60=0$: and a like result would be obtained from the

same time from two points, C and D, and travel in the straight line passing through those points, A at the rate of a miles, and B at the rate of b miles, an hour. How far will each travel before they shall be together, CD being given $=c$?



For solving this problem, let E be the point where the couriers, travelling in the direction CDE, will be together; and put CE $=x$, and DE $=y$. Then we have $x-y=c$. Also, since A travels a miles an hour, if the time before they come together were multiplied by a, the product would be x; and, conversely, $\frac{x}{a}$ would be the time. For the same reason, $\frac{y}{b}$ would also be the same time. Hence $\frac{x}{a}=\frac{y}{b}$; and consequently $bx=ay$. From the resolution of this equation and the former, $x-y=c$, we get $x=\frac{ac}{a-b}$, and $y=\frac{bc}{a-b}$.

As particular examples, let the distance $c=$CD, be 10 miles, and let $a=7$ and $b=6$. Then, from the foregoing expressions, we get $x=70$ and $y=60$ miles, the spaces travelled by A and B respectively before being together. If a had been $=15$ and $b=14$, we should have had $x=150$ and $y=140$ miles. If, again, a were $=10$ and $b=9{\cdot}99$, we should have $x=10000$, and $y=9990$ miles; the spaces to be travelled before the one should overtake the other, the spaces being always greater, the less the velocities of the travellers differ.

Suppose now $a=b$; that is, that the rates of travelling are equal: then $x=\infty$, and $y=\infty$; whence it appears, that they can never come together at any distance, however great.

Suppose, again, $a=10$, and $b=11$: then $x=-100$, and $y=-110$. To interpret the meaning of these negative results, it is plain, in the first place, that the couriers can never come together on the side towards E; since, because B travels more

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second equation. This would be illustrated still better, by using for x and y their uncontracted values,

$$x=\frac{25\times 9-20\times 11}{12\times 25-20\times 15}, \text{ and } y=\frac{12\times 11-15\times 9}{12\times 25-20\times 15},$$

and then multiplying by the denominators, still unincorporated; as it would be found that the members of the results would be identical.

rapidly than A, their distance asunder would be continually increasing. If, however, they started from C and D, as before, at the rates we have supposed, but travelled in the opposite direction, they would be together at a point E′, 100 miles from C, and 110 from D, as would appear by interchanging a and b in the values of x and y. Keeping up the supposition, however, that they are moving in the direction CDE, we shall readily see, that had they been travelling at the same rates *before* arriving at the points C and D, they must have been together at a point E′, 100 miles behind C, and 110 behind D; and, in travelling through this space, B would have gained 10 miles on A, so that they would be simultaneously at C and D. The true interpretation, therefore, of the negative results, -100 and -110, is, that they denote lines or distances, lying in a direction opposite to that contemplated in the solution: and this will be found to be the true mode of interpretation in all similar cases.

If, again, $a=3$, and $b=-2$, we get $x=6$, and $y=-4$. Here, -2 denotes a supposition contrary to that adopted in the solution, signifying that B travels in the direction DCE′: and the result, -4, corresponds to this new supposition, showing that the point of meeting will be between C and D, and 4 miles to the left of D.

In the last place, if c $(=\text{CD})=0$, so that D may coincide with C, the values of x and y, with one exception immediately to be noticed, will be each equal to nothing; which shows, what is otherwise evident, that the couriers will be together at C, and only there. The exception referred to is the case in which a and b are equal. In that case, the values of x and y become vanishing fractions, the terms of which have no common vanishing factor. Hence (§ 129.) these values and the question itself are indeterminate. It is plain, indeed, that the couriers will be always together, however far they travel.

Exam. 16. If a cask be supplied by three cocks, which can fill it in a minutes, b minutes, and c minutes, respectively; in what time would it be filled, if they were all opened at once?

To solve this question, let x be the required time: then, the whole cask being taken as the unit, the parts of it filled by the three cocks in 1 minute, 2 minutes, ..., x minutes, will evidently be severally as follows:

$$\frac{1}{a}, \frac{2}{a}, \ldots, \frac{x}{a}; \quad \frac{1}{b}, \frac{2}{b}, \ldots \frac{x}{b}; \quad \frac{1}{c}, \frac{2}{c}, \ldots, \frac{x}{c};$$

and, as the whole cask is filled in the time x, the parts filled by the three cocks must be equal to 1, the whole cask. Hence we have equation (1.), the resolution of which gives x. We derive (2.) from (1.), by multiplying by abc, to remove the denominators; and (3.) is the same as (2.), the form of the first member (§ 51.), and the order of the terms being changed. Then the value of x is found by division.

$$\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1 \ldots . (1.)$$

$$bcx+acx+abx=abc \ldots . (2.)$$

$$\text{or, } (ab+ac+bc)x=abc \ldots . (3.)$$

$$x=\frac{abc}{ab+ac+bc} \ldots . (4.)$$

If, as a particular example, $a=10$, $b=20$, and $c=30$, x will be found to be $5\frac{5}{11}$ minutes.

Suppose, again, $a=20$, $b=30$, and $c=\infty$. In this case, the two terms, ac and bc, in the denominator of (4.), becoming infinite, the finite term ab will be infinitely small in comparison of them, and may therefore be rejected. Then, by dividing the numerator and denominator by c, we get $x=\frac{ab}{a+b}=12$ minutes. It is better, however, to divide the numerator and denominator of (4.) by c. Then, (§ 127.) c being infinite, the term $\frac{ab}{c}$ in the denominator vanishes, and we get the same result as before.

The interpretation here is plain. If c be a very long time, the third cock must introduce the fluid very slowly: and, if it be infinite, the meaning is, that no fluid is introduced by that cock; since, if it introduced even the smallest quantity, it would fill the cask in some finite time. The question, therefore, is thus changed into one regarding *two cocks* instead of three.

If, again, $a=10$, $b=20$, and $c=-30$, we get $x=8\frac{4}{7}$. The meaning here is, that the third cock draws off fluid, instead of introducing it. This will be plain from equation (1.), in which, when c is negative, the third term, which is the effect produced by the third cock, is negative, and thus it is subtracted from the sum of the first and second terms, which is the joint effect produced by the other two cocks.

If a were $=20$, $b=30$, and $c=-12$, x would be found to be infinite; as, in that case, the quantity carried off by the third cock would be exactly equal to what would be introduced by the other two.

In the last place, if $a=20$, $b=30$, and $c=-10$, we should find $x=-60$; a result which shows that, in this case, there is an incompatibility in the conditions of the question. The problem would then be virtually reduced to the following. If a vessel be supplied with two cocks, one of which would fill it in 20 minutes, and the other in 30, and with a third which can empty it in 10 minutes; in what time, *if it were empty, would it be filled*, if all the cocks were opened. Now this enunciation is absurd; as the first and second requiring, as we have already seen, 12 minutes to fill it, cannot introduce so much as would be carried off by the third; and the value, -60, shows that, if it had been full 60 minutes before, it would now have been emptied by the action of all the three cocks. The question would be rendered correct by changing the words in italics into the following; *if it were full, would it be emptied.**

Exam. 17. Find a number expressed in the decimal notation by two digits, whose sum is 10; and such, that if 1 be taken from its double, the remainder will be expressed by the same digits in a reversed order.

Let x and y be the digits. Then, by the nature of the decimal notation, the number will be $10x+y$; and the other number expressed by the same digits will be $10y+x$. Hence, by the question, we shall have the two equations (1.) and (2.) in the margin. We find equation (3.) from (2.) by transposition: and from (1.) and (3.), we readily find, by any of the ordinary methods, $x=3$, and $y=7$: so that the required number is 37. This answers the conditions of the question, as its double is 74; and, if 1 be taken from this, the remainder 73 is expressed by the same digits as 37, but in a reversed order, and having their sum $=10$.

$$x+y=10 \; \ldots\ldots\ldots (1.)$$
$$20x+2y-1=10y+x \; \ldots\ldots (2.)$$
$$19x-8y=1 \; \ldots\ldots\ldots (3.)$$

Had there been three digits, the number would have been expressed by $100x+10y+z$: and, in general, any number would be expressed in the decimal notation by

$$10^{n-1}x_1+10^{n-2}x_2+\ldots+10x_{n-1}+x_n,$$

where n is a whole number, and x_1, x_2, x_3, &c., are each of them any one of the digits, 1, 2, 3, ..., 9, or the cipher, 0.

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* The student will find it easy to generalise the views here given regarding the foregoing question, by using a, b, and c, instead of the particular values employed above.

Exercises.

30. A cask contains a certain number of gallons of rum, and an mth part of that quantity of water: but, if a gallons of rum and b of water be added to the mixture, the water in the whole compound will be an nth part of the rum. Required the quantity of each contained in the cask at first. Examine also and explain the case, in which, m being equal to n, a is equal to nb, and the one in which it is not equal to it; and also the case in which x and y come out negative, x denoting the original number of gallons of the rum, and y those of the water.

Ans. $y = \dfrac{nb-a}{m-n}$, and $x = \dfrac{m(nb-a)}{m-n}$. When $m=n$, and $a=nb$, the question is indeterminate: when $m=n$, and $a >$ or $< nb$, the values of x and y are infinite, and the question absurd. When x and y are negative, the question will be changed into one in which the quantity of rum is *diminished* by a gallons, and that of the water by b gallons.

31. Find a fraction, such that if its denominator be increased by 1, the value becomes $\frac{1}{2}$; while if the numerator be increased by 1, the value is $\frac{3}{5}$. *Ans.* $\frac{8}{15}$.

32. Required a fraction, such that if the numerator and denominator be each increased by 1, the value is changed into $\frac{1}{2}$; but, if they be each diminished by 1, the value becomes $\frac{1}{3}$. *Ans.* $\frac{3}{7}$.

33. One person says to another: "If you give me half your money, I shall have a hundred pounds." The other replies: "I shall have a hundred pounds, if you give me a third of your money." How much had each? *Ans.* £60 and £80.

34. Given the mean of three equidifferent numbers $= a\ (=20)$, and the product of the extremes $= b\ (=279)$; to find the extremes. *Ans.* $a-\sqrt{(a^2-b)}\ (=9)$, and $a+\sqrt{(a^2-b)}\ (=31)$.

35. Find the extremes of three numbers in arithmetical progression, the mean being $= a\ (=11)$, and the sum of the squares of the extremes $= b\ (=260)$: and find the extremes of three others, such that the mean may be $= a\ (=13)$, and the difference of the squares of the extremes $= b\ (=416)$.

Ans. The first pair, $a-\sqrt{(\frac{1}{2}b-a^2)}\ (=8)$, and $a+\sqrt{(\frac{1}{2}b-a^2)}\ (=14)$; and the second pair, $a-\dfrac{b}{4a}\ (=5)$, and $a+\dfrac{b}{4a}\ (=21)$.

36. Find four equidifferent numbers, such that the sum of

the squares of the extremes may be $= a\ (=410)$, and the sum of the squares of the means $= b\ (=346)$.

Ans. $\frac{1}{4}\{\sqrt{(9b-a)}-3\sqrt{(a-b)}\}\ (=7)$,
$\frac{1}{4}\{\sqrt{(9b-a)}-\sqrt{(a-b)}\}\ (=11)$,
$\frac{1}{4}\{\sqrt{(9b-a)}+\sqrt{(a-b)}\}\ (=15)$, and
$\frac{1}{4}\{\sqrt{(9b-a)}+3\sqrt{(a-b)}\}\ (=19)$.

37. Given the difference of the squares of the extremes of four equidifferent numbers $= a$, and the difference of the squares of the means $= b$: to find the numbers.

Ans. The question is indeterminate, if $a = 3b$; otherwise it is absurd: the one difference being always treble of the other.

38. Given the sum of four equidifferent numbers $= a\ (=26)$, and the sum of their squares $= b\ (=214)$; to find them.

$$\textit{Ans. } \tfrac{1}{4}\left(a-3\sqrt{\frac{4b-a^2}{5}}\right)\ (=2),$$

$$\tfrac{1}{4}\left(a-\sqrt{\frac{4b-a^2}{5}}\right)\ (=5),$$

$$\tfrac{1}{4}\left(a+\sqrt{\frac{4b-a^2}{5}}\right)\ (=8), \text{ and}$$

$$\tfrac{1}{4}\left(a+3\sqrt{\frac{4b-a^2}{5}}\right)\ (=11).$$

39. Given the difference of the squares of the first and third of four equidifferent numbers $= a\ (=56)$, and the difference of the squares of the second and fourth $= b\ (=40)$; to find the numbers.

$$\textit{Ans. } \frac{2a-b}{2\sqrt{(a-b)}}\ (=9), \quad \frac{a}{2\sqrt{(a-b)}}\ (=7),$$

$$\frac{b}{2\sqrt{(a-b)}}\ (=5), \text{ and } \frac{2b-a}{2\sqrt{(a-b)}}\ (=3).$$

40. At what time, between ten and twelve o'clock, are the hour and minute hands of a common clock exactly together.

Ans. At $5\frac{5}{11}$ minutes before eleven.

41. Find two numbers, such that one third of the first exceeds one fourth of the second by 3, and that one fourth of the first and one fifth of the second are together equal to 10. *Ans.* 24 and 20.

42. Required two numbers, such that the sum of one half of the first and one third of the second may be 29, and that one third of the first and one fourth of the second may amount to 21.

Ans. 18 and 60.

43. A number expressed by three digits, whose sum is 22, is

less by 297 than the number expressed by the same digits in a reversed order, and its first digit is less by 1 than its second. What is the number? *Ans.* 679.

44. A bill of 100*l.* may be paid by 50 bank notes of one value each, and by 38 of another; or it may be paid by means of 75 of the former kind, and 17 of the latter. What are the values of the notes? *Ans.* Those of the first kind 21 shillings each, and those of the second 25 shillings.

45. Two persons set out from a certain place, on the same day, and proceed in the same direction, the one travelling 30 miles the first day, and going each day a mile less than he did on the preceding; while the other travels at the constant rate of 20 miles a day. When will they next be together?

Ans. At the end of 21 days.

46. How many lines are contained in a page of a book, and how many letters at an average in each line of that page, if it be found that by adding one line to each page, and making each line contain an additional letter, the page will be increased by 96 letters; while, by adding two lines to the original page, and making each line contain four additional letters, the number of letters will be increased by 286?

Ans. 44 lines, each containing 51 letters.

47. Two persons get each a legacy of £300, and one of them is then found to be worth three times as much as the other: but had the legacy to each been £800, the one would have been worth only twice as much as the other. How much had each originally?

Ans. £1200 and £200.

CHAPTER IX.

QUADRATIC EQUATIONS.

150. Pure equations of the second or any higher degree are resolved in the same manner, in every respect, as equations of the first degree, except that, at the conclusion, the root corresponding to the degree of the equation is to be extracted.

Thus, if $7x^2-4=4x^2+23$, we get, by transposition and contraction, and by dividing by 3, $x^2=9$; whence by extracting the square root, we find $x=\pm 3$. Such equations have already been resolved in a few instances of a simple kind, such as in Exam. 11. p. 90.

151. The general form in which a compound quadratic equation may always be exhibited is $ax^2+bx=c$, where a, b, and c are known or given quantities, and x is the quantity to be found: and we may now proceed, as in the margin, to investigate the method of determining that quantity.

$$ax^2+bx=c \;\ldots\ldots\ldots\; (1.)$$
$$4a^2x^2+4abx=4ac \;\ldots\ldots\; (2.)$$
$$4a^2x^2+4abx+b^2=b^2+4ac \;\ldots\; (3.)$$
$$2ax+b=\pm\sqrt{(b^2+4ac)} \;\ldots\; (4.)$$
$$x=\frac{-b\pm\sqrt{(b^2+4ac)}}{2a} \;\ldots\; (5.)$$

In this process, equation (2.) is found from (1.) by multiplying by $4a$; and (3.) from (2.), by adding b^2 to both members. We then find (4.) from (3.) by extracting the square root, according to § 53. and 119.; and (5.) from (4.), by transposing b and dividing by $2a$.

By examining the expression thus found for x, in connexion with the given equation (1.), we see that the first term of the numerator is the coefficient of the second term of (1.) with its sign changed: that this is followed by the double sign ±, and by the sign of the square root: that the quantity affected by the latter sign consists of two parts, the first b^2, which is the square of the coefficient b, and the second $4ac$, which is four times the product of a, the coefficient of the higher power of x, and c, the absolute term of the equation, that is, the term which does not contain x. Hence we have the following general rule.

To resolve an equation of the second degree; reduce the given equation, by transposition or other operations, to the form $ax^2+bx=c$, if it be not of that form already. Then, to find x, to the coefficient of the second term, with its sign changed, annex, by the double sign ±, the square root of the quantity obtained by adding together the square of that coefficient, and four times the product of the absolute term and the coefficient of the first term; and, lastly, divide the whole expression thus obtained by twice this last-mentioned coefficient.

Exam. 1.

$$x^2+2x-9=\frac{x^2+3x}{3} \;\ldots\; (1.)$$
$$3x^2+6x-27=x^2+3x \;\ldots\; (2.)$$
$$2x^2+3x=27 \;\ldots\ldots\; (3.)$$
$$x=\frac{-3\pm\sqrt{(9+216)}}{4} \;\ldots\; (4.)$$
$$x=\frac{-3\pm15}{4} \;\ldots\ldots\ldots\; (5.)$$
$$x=3,\text{ and }x=-4\tfrac{1}{2} \;\ldots\ldots\; (6.)$$

In this example we derive (2.) from (1.) by multiplying by 3, so as to remove the denominator. We then obtain (3.) from (2.) by transposing x^2, $3x$, and -27, and contracting the result: and thus the preparatory operations are completed. Then, to resolve the equation thus obtained, we write down x and the sign of equality; and to find the expression equivalent to x, we write in succession, in the numerator, -3, the second coefficient, with its sign changed; the double sign $\pm$, and the sign of the square root: after this sign, we place, in a vinculum, the square of the coefficient 3, together with 216, which is four times the product of the coefficient 2 and the absolute term 27: and, lastly, we write as denominator 4, the double of the first coefficient; thus finishing the solution, except the numerical operation. To effect this, we commence, as we ought always to do, with the radical, which by addition becomes $\pm\sqrt{225}$, or, by actual extraction, ± 15; and we thus get equation (5.). Now, as 15 is preceded not only by the sign $+$, but also by $-$, we must take it first with the one and then with the other. If $+$ be taken, the numerator becomes $-3+15$, or 12; and dividing this by the denominator 4, we get $x=3$: but if $-$ be taken, the numerator will be $-3-15$, or -18; and, by dividing this by the denominator, we find $x=-4\frac{1}{2}$.

It thus appears, that x has two distinct values; each of which will be found to satisfy the given equation. Thus if $x=3$, we have $x^2=9$, and the first member of (1.) will become $9+6-9$, or, by contraction 6; while the second will become $\frac{9+9}{3}$, which, by contraction, also becomes 6: so that when $x=3$, the equation is satisfied, its two members being rendered equal. Again, if $x=-4\frac{1}{2}$, or $-\frac{9}{2}$, we have $x^2=\frac{81}{4}$, or $20\frac{1}{4}$; and the first member of (1.) becomes $20\frac{1}{4}-9-9$, or $2\frac{1}{4}$; while the second becomes $\frac{20\frac{1}{4}-13\frac{1}{2}}{3}$; or, by contraction, $2\frac{1}{4}$, the same as the first member; so that this value of x likewise satisfies the equation. It is plain, also, that equation (4.) or (5.) can give no other value for x. *The unknown quantity x, therefore, has two values or roots, and only two; and this is universally the case;* as is evident from the general expression (5.) in § 151., which will give one value, if the radical be taken positive, and another if negative; but it will admit of no other variety in its results. The relations and properties of the roots of quadratic equations will form a future and separate subject of consideration. In the mean time the student should make himself master of the mode of finding the roots; and he ought frequently to verify his results by substituting them in the given equation.

Exam. 2.

$$\frac{8x^2+5x}{5}=x^2+2x+10 \quad \ldots\ldots\ldots\ldots (1.)$$

$$8x^2+5x=5x^2+10x+50 \quad \ldots\ldots\ldots (2.)$$

$$3x^2-5x=50 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

$$x=\frac{5\pm\sqrt{(25+4\times3\times50)}}{6} \quad \ldots (4.)$$

$$x=\frac{5\pm25}{6} \quad \ldots\ldots\ldots\ldots\ldots (5.)$$

$$x=5, \text{ and } x=-3\tfrac{1}{3}.$$

Here (2.) is found from (1.) by multiplying by 5, and (3.) from (2.) by transposing $5x^2$ and $10x$, and contracting; and this finishes the preparatory operations. To find (4.), we write down x, followed by the sign of equality; and, to get the second member, we write in succession, as forming the numerator, 5 (the second coefficient with its sign changed), the double sign, and the sign of the square root; after which we place in a vinculum the sum of the square of 5, and four times the product of the first coefficient 3, and the absolute term 50; and, lastly, we write the double of the first coefficient as denominator. For the numerical operation we contract the radical, which thus becomes $\pm\sqrt{625}$, or, by extraction, ± 25; and thus we get (4.). If 25 be taken positive, the numerator becomes $5+25$, or 30; and dividing this by 6, we find $x=5$: but, if 25 be taken negative, the numerator becomes $5-25$, or -20; and, dividing this by 6, we get $-3\frac{1}{3}$ for the other value of x: and this value, as well as the foregoing, will be found to satisfy the equation.

Exam. 3.

$$x^2+6=6x-\frac{3x+2}{7} \quad \ldots\ldots\ldots\ldots (1.)$$

$$7x^2+42=42x-3x-2 \quad \ldots\ldots\ldots (2.)$$

$$7x^2-39x=-44 \quad \ldots\ldots\ldots\ldots\ldots (3.)$$

$$x=\frac{39\pm\sqrt{(39^2-4\times7\times44)}}{14} \quad \ldots (4.)$$

$$x=\frac{39\pm17}{14} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

$$x=4, \text{ and } x=1\tfrac{4}{7}.$$

In this example, by multiplying by 7 we get (2.), and thence, by transposition and by contracting, we obtain (3.). Then, for the numerator of the value of x in (4.), we write in succession 39, the second coefficient, with its sign changed; the signs $\pm$ and $\sqrt{}$; and after the latter we place, in a vinculum, the square of 39, and $-4\times7\times44$, four times the product of the absolute term, and the first coefficient; and the denominator 14 is the double of this coefficient. In computing the radical, we find the square of 39, which is 1521; and the continual product of 4, 7, and 44 is found to be 1232. This, as the sign directs, is taken from 1521; and, the remainder being 289, its square root is extracted, and is found to be 17. We thus get (5.), and from it we find $x=4$, and $x=1\frac{4}{7}$, each of which answers. The student will observe that, contrary to what was the case in the two preceding examples, the roots here are both positive.

Exam. 4.

$$x-1=\sqrt{(x+1)}+4 \quad \ldots\ldots\ldots\ldots (1.)$$
$$x-5=\sqrt{(x+1)} \quad \ldots\ldots\ldots\ldots (2.)$$
$$x^2-10x+25=x+1 \quad \ldots\ldots\ldots\ldots (3.)$$
$$x^2-11x=-24 \quad \ldots\ldots\ldots\ldots (4.)$$
$$x=\frac{11\pm\sqrt{(121-96)}}{2} \quad \ldots\ldots\ldots\ldots (5.)$$
$$x=\frac{11\pm5}{2} \quad \ldots\ldots\ldots\ldots (6.)$$
$$x=8,\text{ and }x=3 \quad \ldots\ldots\ldots\ldots (7.)$$

Here we first clear the equation of the radical, by the method pointed out in § 110. To effect this, we transpose 4 in (1.), and thus we obtain (2.); and, by squaring the members of this, we get (3.); from which, by transposing 25 and x, we derive (4.). Then, by resolving this by § 151. we get (5.), and thence, successively, (6.) and (7.).

If, for the sake of verification, we substitute 8 for x, we get the first and second members each equal to 7, so that the value 8 satisfies the equation. Substituting 3, however, we have apparently the first member equal to 2 and the second to 6. The interpretation here is simply this, that in (1.) the term $\sqrt{(x+1)}$ may have either + or − prefixed to it, as the case may require; so that equation (1.) might be written $x-1=4\pm\sqrt{(x+1)}$, 8 being the value that must be given to x, when the upper sign is used; while 3 is the value corresponding to the lower.*

The work here proceeds in the same manner as in the preceding examples, after the preparatory operations were performed in each; and the answers are found to be 2 and $-\frac{4}{13}$. When, however, as in this example, the coefficient of the second term is even, the solution may be obtained more easily by means of a rule, which may now be investigated.

Exam. 5.

$$13x^2-22x=8 \quad \ldots\ldots\ldots\ldots (1.)$$
$$x=\frac{22\pm\sqrt{(484+416)}}{26} \quad \ldots\ldots (2.)$$
$$x=\frac{22\pm30}{26} \quad \ldots\ldots\ldots\ldots (3.)$$
$$x=2,\text{ and }x=-\tfrac{4}{13} \quad \ldots\ldots (4.)$$

---

* Most of the succeeding examples will be assumed so as to require no preparatory work, as that can in general present no difficulty.

152. Let the equation to be resolved be $ax^2+2bx=c$, where the coefficient $2b$ is divisible by 2.

Here (2.) is derived from the given equation by multiplying by a, and (3.) from (2.) by adding b^2. Then (4.) is obtained from (3.) by extracting the square root, and (5.) from (4.) by transposing b and dividing by a. By comparing this result with the given equation, and expressing it in words, we have the following rule:

$$ax^2+2bx=c \quad \ldots\ldots\ldots\ldots (1.)$$
$$a^2x^2+2abx=ac \quad \ldots\ldots\ldots (2.)$$
$$a^2x^2+2abx+b^2=b^2+ac \quad \ldots (3.)$$
$$ax+b=\pm\sqrt{(b^2+ac)} \quad \ldots\ldots (4.)$$
$$x=\frac{-b\pm\sqrt{(b^2+ac)}}{a} \quad \ldots\ldots (5.)$$

To resolve an equation of the second degree in the easiest manner, when the coefficient of the second term is even; to half that coefficient with its sign changed, annex, by the sign $\pm$, the square root of the sum obtained by adding together the square of that half coefficient, and the product of the absolute term by the first coefficient, and divide the result by the last-mentioned coefficient.

The work of the last example by this rule will be as in the margin. The second member of equation (2.) is found from (1.) by writing 11, half the second coefficient with its sign changed; by placing after it the signs $\pm$ and $\surd$; and by subjoining to the latter, in a vinculum, the square of 11, together with the product of 8, the absolute term, and 13, the first coefficient: and, finally, by dividing the result by this coefficient. The mode of performing the rest of the work is obvious: and it will be seen, that the facility obtained in this method consists simply in employing smaller numbers than those required in using the first rule. Thus, by comparing the two solutions, we find that 11, 13, and 15, in the latter method, are respectively the halves of the corresponding numbers in the former; while 121 and 104, the numbers in the one radical, are each only a fourth part of those in the other.

$$13x^2-22x=8 \quad \ldots\ldots\ldots\ldots (1.)$$
$$x=\frac{11\pm\sqrt{(121+104)}}{13} \quad \ldots (2)$$
$$x=\frac{11\pm15}{13} \quad \ldots\ldots\ldots\ldots (3.)$$
$$x=2, \text{ and } x=-\tfrac{4}{13} \quad \ldots\ldots (4.)$$

It may be remarked that each of the foregoing rules will resolve *any* quadratic equation whatever; but that, in practice, the latter ought to be used when the second coefficient is even, and the former when it is odd.

Exam. 6. Given the sum of two numbers = 18, and their product = 77; to find them.

Here, let x be one of the numbers; then $18-x$ will be the other. Taking the product of these, we obtain $18x-x^2$, which by the question is to be equal to 77; and we thus get equation (1.). Then (2.) is obtained from (1.) by changing all the signs, and (3.) from (2.) by the second rule. In this operation, since, in equation (2.), the coefficient of x^2 is 1, it is unnecessary to divide by it in (3.); and the same is the case in all similar instances. We finally get $x=11$, and $x=7$. If we take the former of these values, we get for the second number $18-11$, or 7; while if we take the latter, we get $18-7$, or 11. Hence the required numbers are 11 and 7, or 7 and 11. In this example, therefore, we have in reality only one distinct solution, as the numbers obtained are the same in both cases, but in a reversed order.*

$$18x-x^2=77 \quad \ldots\ldots\ldots (1.)$$
$$x^2-18x=-77 \quad \ldots\ldots (2.)$$
$$x=9\pm\sqrt{(81-77)} \ldots (3.)$$
$$x=9\pm 2 \quad \ldots\ldots\ldots\ldots (4.)$$
$$x=11, \text{ and } x=7 \quad \ldots\ldots (5.)$$

---

* This circumstance (that of two values being obtained for the unknown quantity, when the question admits of only one solution,) takes place, in the present and similar instances, where the two required quantities are employed in the question in exactly the same manner. Thus, in the present question, the two are added together to find 18, and are multiplied together to find 77. In the solution, x was assumed to denote one of the parts of 18; and there was nothing either in that assumption or in the process, to make it denote one part in preference to the other. The operation, therefore, gives them both, as it ought.

It may be remarked, that the two rules might be readily modified, so as to give the values of x directly from equation (1.), without changing the signs so as to find (2.). Algebraists, however, universally write equations, when prepared for resolution, so that the highest power of the unknown quantity may stand first, and may be positive; and this uniformity of arrangement is attended with advantage.

Example 6. may also be solved in a neat manner, as in the margin, by means of two unknown quantities, and without employing either of the rules given in § 151. and § 152. In this process, (3.) is obtained from (1.) by squaring both members, and (4.) from (2.) by multiplying by 4. Then (5.) is found from (3.) and (4.) by subtraction; and the first member of this being (§ 53.) the square of $x-y$ (or of $y-x$), we get (6.) by taking the square roots of both members. By (1.), therefore, we have

$$x+y=18 \ldots\ldots\ldots\ldots (1.)$$
$$xy=77 \ldots\ldots\ldots\ldots\ldots (2.)$$
$$x^2+2xy+y^2=324 \ldots\ldots (3.)$$
$$4xy=308 \ldots\ldots\ldots (4.)$$
$$x^2-2xy+y^2=16 \ldots\ldots (5.)$$
$$x-y=4 \ldots\ldots\ldots\ldots (6.)$$
$$x=11, \text{ and } x=7 \ldots\ldots (7.)$$

To give a general solution of this problem, let s denote the sum and p the product of the given numbers, and let x be one of them: then $s-x$ will be the other; and the work will be as in the margin. From (3.) it appears that to find the numbers, we are to subtract four times the given product from the square of the given sum, and to take the square root of the remainder. Then we are to take the sum and difference of that root and the given sum, and to divide the results by 2.

$$x(s-x)=p \ldots\ldots\ldots (1.)$$
$$x^2-sx=-p \ldots\ldots\ldots (2.)$$
$$x=\frac{s\pm\sqrt{(s^2-4p)}}{2} \ldots (3.)$$

Thus, if the sum were 17, and the product 60, we should have the square of 17 equal to 289, and four times 60 equal to 240. Taking the latter of these from the former, we should get 49, the square root of which is 7. Then $17+7=24$, and $17-7=10$; the halves of which are 12 and 5, the required numbers.

If, again, the sum of two numbers be given equal to 13, and their product equal to 50, we should have, by equation (3.),

$$x=\frac{13\pm\sqrt{(169-200)}}{2}=\frac{13\pm\sqrt{-31}}{2}.$$

Now, since (§ 121.) -31 has no square root, there can be no solution to this question. The data, in fact, are inconsistent with one another; as it is impossible that there can be two numbers whose sum is 13 and product 50. Equation (3.) contains the radical $\sqrt{(s^2-4p)}$; and this will always be imaginary, if $4p$ be

the sum, and by (6.) the difference, of x and y: and these quantities are then found by § 52.

This question may also be solved by means of a pure quadratic, by putting half the difference of the numbers $=x$. Then, since half their sum is 9, we shall have (§ 52.) the numbers themselves expressed by $9+x$ and $9-x$; and, since the product of these is to be 77, we get at once equation (1.). From this we derive (2.) by actual multiplication (§ 57.), and by changing the signs; and thence by transposition, &c., we get $x=2$. Then, half the sum being 9 and half the difference 2, we get the numbers by means of § 52.

$$(9+x)(9-x)=77 \ldots\ldots (1.)$$
$$x^2-81=-77 \ldots\ldots (2.)$$
$$x^2=4 \ldots\ldots\ldots\ldots (3.)$$
$$x=2 \ldots\ldots\ldots\ldots (4.)$$

Had the difference, and consequently the half-difference been given, we might with equal advantage have put x to denote half the sum. We have thus two instances, (and there are numberless others,) in which *it is advantageous to find, instead of the quantity required, another which can be obtained more easily, and from which the one required can be derived.*

greater than s^2. The greatest possible value, therefore, that p can have is $\frac{1}{4}s^2$, or, in the present instance, $42\frac{1}{4}$; and, consequently, when the product is given $= 50$, the question is absurd. By means of this principle we have it in our power, in many instances, to ascertain whether problems are possible or impossible, and also to determine the greatest or least values that quantities can have in particular cases.

Exam. 7. Given the sum of the side and diagonal of a square $= a$; to find the side.

Let x be the side: then (Euc. I. 47.) the diagonal will be $\sqrt{2x^2}$; and adding x to this, we get equation (1.). Then equation (2.) is obtained by transposing x, and (3.) by squaring. From this we get (4.) by transposing and contracting:

$$\sqrt{2x^2}+x=a \ldots\ldots\ldots (1.)$$
$$\sqrt{2x^2}=a-x \ldots\ldots\ldots (2.)$$
$$2x^2=a^2-2ax+x^2 \ldots\ldots (3.)$$
$$x^2+2ax=a^2 \ldots\ldots\ldots (4.)$$
$$x=-a\pm\sqrt{(a^2+a^2)} \ldots (5.)$$
$$x=-a\pm a\sqrt{2} \ldots\ldots\ldots (6.)$$

and from it (5.) is obtained by § 152.: and (6.) is the same as (5.) contracted, and modified by § 101. Were we here to take the lower sign, so that $a\sqrt{2}$ would be negative, the value of x would be negative; and therefore, for giving a solution in the plain arithmetical sense of the problem, this term must be taken positive. Hence, suppose the given sum to be 100 inches, and from (6.) we shall have the side,

$$x=(\sqrt{2}-1)\times 100=(1{\cdot}4142136, \&c.-1)\times 100=41{\cdot}42136,$$ &c.

This question may be solved very easily, as in the margin; where equation (2.) is the same as (1.) modified by § 101.; and (3.) the same as (2.) modified by § 51. Equation (4.) is obtained from (3.) by division; and (5.) is derived from (4.) by § 108.

$$\sqrt{2x^2}+x=a \ldots\ldots (1.)$$
$$x\sqrt{2}+x=a \ldots\ldots (2.)$$
$$(\sqrt{2}+1)x=a \ldots\ldots (3.)$$
$$x=\frac{a}{\sqrt{2}+1} \ldots\ldots (4.)$$
$$x=(\sqrt{2}-1)a \ldots\ldots (5.)$$

153. While, as we have seen in Exam. 1., negative values of the unknown quantity will satisfy the equation from which they are derived, any such value is inadmissible as a solution of the question of which that equation is the algebraic expression, when the enunciation of the question is taken in its plain, ordinary meaning. It is an important fact, however, that in quadratic equations, and, in many instances, in others, every such value,

with its sign changed, is a solution of a problem analogous to the original one; differing from it merely in this, that certain quantities are added instead of being subtracted, or are subtracted instead of being added. Thus, in the equation, $ax^2+bx=c$, if we change the sign of x, we get $a(-x)^2-bx=c$, or $ax^2-bx=c$, which is the same as the original equation, except that the term bx is to be subtracted instead of being added. In a similar way, $ax^2-bx=c$ becomes $a(-x)^2-b\times -x=c$, or $ax^2+bx=c$; where the term bx is to be added instead of being subtracted.

The last example affords an illustration of this principle. In it the negative root was $-a-a\sqrt{2}$; which, if its signs be changed, becomes $a+a\sqrt{2}$. To find the corresponding problem to which this is the solution, we change the sign of x in equation (1.). By this means, we get $\sqrt{2x^2}-x=a$; an expression which is plainly the algebraic translation of the following problem: — *Given the difference of the side and diagonal of a square* $= a$, *to find the side;* and the length of the side will be $a+a\sqrt{2}$, the quantity obtained above. The student will find it useful to work the question just enunciated, by putting x to denote the side, and proceeding as in Exam. 7. In doing so, he will find the positive root to be what is here stated; while the other, with its sign changed, will be the positive root of Exam. 7.

Exam. 8. A person bought a number of yards of cloth for a shillings; and he finds that, if he had got b yards more for the same money, each yard would have cost him c shillings less. How many yards did he purchase?

To solve this, let x be the number of yards: then each yard must have cost $\frac{a}{x}$ shillings; but, if he had got $x+b$ yards for the same money, each yard would have cost him $\frac{a}{x+b}$ shillings, which, by the question is less by c than the actual price of each yard, as found above; and hence we have equation (1.). By clearing this of fractions and rejecting ax, we get (2.); and from this (3.) is ob-

$$\frac{a}{x+b}+c=\frac{a}{x} \quad \dots\dots\dots\dots\dots (1.)$$

$$cx^2+bcx=ab \quad \dots\dots\dots\dots\dots (2.)$$

$$x=\frac{-bc\pm\sqrt{(b^2c^2+4abc)}}{2c}{}^{*} \quad \dots\dots (3.)$$

---

[*] The numerator and denominator of this fraction are divisible by $\sqrt{c}$. For the purposes of computation, however, it is better to retain it as it is.

tained by § 151. As a particular example, let $a=60$, $b=1$, and $c=2$; and x will be found from (3.) to be 5, or -6. The former of these is the answer to the proposed question in its ordinary meaning. To find the general problem to which the negative value of x, with its sign changed, is the answer, we have merely to change x into $-x$ in equation (1.): then, by changing the signs of all the terms in the result, we get $\frac{a}{x-b}-c=\frac{a}{x}$; an equation which is evidently the algebraic expression of the following question: —

A person bought a certain number of yards of cloth for a shillings; and he finds, that if he had got b yards fewer for the same money, each yard would have cost c shillings more. How many yards did he purchase?

Resolving the equation just found we should get for x the same expression as in equation (3.), except that the first term of the numerator would be bc not $-bc$; and taking, as above, $a=60$, $b=1$, and $c=2$, we should have $x=6$, and $x=-5$, the former of which is the answer to the new question in its obvious meaning.*

Exam. 9. Given one leg of a right-angled triangle $=12$ inches, and the excess of twice the hypotenuse above the remaining leg $=21$ inches; to find that leg and the hypotenuse.

Let x be put to denote the remaining leg; then (Euc. I. 47.) the hypotenuse will be $\sqrt{(x^2+144)}$; and, by the question, we have equation (1.). From this we get (2.) by squaring; and (3.) is obtained from (2.) by transposing and contracting. Equation (4.) is derived from this by dividing by 3 for a simplification; and (5.)

$$2\sqrt{(x^2+144)}=x+21 \quad \ldots\ldots\ldots (1.)$$
$$4x^2+576=x^2+42x+441 \quad \ldots\ldots (2.)$$
$$3x^2-42x=-135 \quad \ldots\ldots\ldots (3.)$$
$$x^2-14x=-45 \quad \ldots\ldots\ldots (4.)$$
$$x=7\pm\sqrt{(49-45)}=7\pm2 \quad \ldots (5.)$$
$$x=9, \text{ and } x=5 \quad \ldots\ldots\ldots (6.)$$

---

* To gain farther improvement in tracing such relations as we have been considering, the student may solve the following question, either deriving its solution from equation (3.), or solving it independently of what is done above. A person bought a number of yards of cloth for a shillings; and he finds that, if he had got b yards fewer for the same money, each yard would have cost him c shillings less. How many yards did he purchase? This question is evidently absurd; and the student may consider how the absurdity is manifested by the algebraic solution. He may also illustrate it in numbers, taking $a=12$, $b=5$, and $c=10$; and also $a=12$, $b=5$, and $c=8$.

is obtained by § 152. It thus appears, therefore, that x may be either 9 or 5. Taking the former, we find the hypotenuse $\sqrt{(x^2+144)}$, to be 15; but taking 5, we get 13 for the hypotenuse. Hence, there are two triangles, entirely distinct and dissimilar, each of which equally answers the question; the sides of one of them being 12, 9, and 15, and those of the other 12, 5, and 13. That each of these answers the question is plain, since each of them has one leg $= 12$, and since twice 15 exceeds 9 in the one by 21, and twice 13 exceeds 5 by the same in the other.

To have a general solution of this problem, let the given leg be denoted by a, and the excess of n times the hypotenuse above the remaining leg by b; and the process will stand as in the margin. In this, (5.) is derived from (4.) by performing the actual multiplication of b^2 by n^2-1, and rejecting b^2-b^2 in the result; and (6.) is obtained from (5.) by means of § 101.

$$n\sqrt{(x^2+a^2)}=x+b \quad \ldots\ldots\ldots\ldots (1.)$$

$$n^2x^2+n^2a^2=x^2+2bx+b^2 \quad \ldots\ldots\ldots (2.)$$

$$(n^2-1)x^2-2bx=b^2-n^2a^2 \quad \ldots\ldots\ldots (3.)$$

$$x=\frac{b\pm\sqrt{\{b^2+(n^2-1)(b^2-n^2a^2)\}}}{n^2-1} \quad \ldots (4.)$$

$$x=\frac{b\pm\sqrt{\{n^2b^2-n^2(n^2-1)a^2\}}}{n^2-1} \quad \ldots\ldots (5.)$$

$$x=\frac{b\pm n\sqrt{\{b^2-(n^2-1)a^2\}}}{n^2-1} \quad \ldots\ldots\ldots (6.)$$

In considering the nature of the results that will be given by this equation according to the relations of the data, we may, throughout, naturally regard the given leg a as positive. Then, if n be positive, it will be greater than 1, or a positive fraction less than 1, or it will be equal to 1. Now, from equation (6.), it will be seen, that the question will be possible only when $(n^2-1)a^2$ does not exceed b^2. Hence, if, as in the foregoing question, n were $= 2$, and $a = 12$, but $b = 20$, instead of 21; these data would be inconsistent with each other, and the question would admit of no solution. If $(n^2-1)a^2$ were equal to b^2, there would be but one solution, as the radical would disappear.

Suppose, in the second place, $n<1$: then n^2-1 would be negative, and therefore $-(n^2-1)a^2$ would be positive. In this case, since b^2 is necessarily positive, the radical would never be imaginary, and the solution would always be possible.

In the third place, suppose $n=1$. Then (6.) becomes

$$x=\frac{b\pm b}{1-1}; \text{ whence } x=\frac{2b}{0}=\infty, \text{ and } x=\frac{0}{0}.$$

Each of these values, if substituted in equation (1.), will satisfy that equation, when $n=1$. Thus, by substituting the former, we get

$$\sqrt{\left(\frac{4b^2}{0^2}+a^2\right)}=\frac{2b}{0}+b\,;$$

and, by multiplying both members by 0, we get $\sqrt{4b^2}$ for the one, and $2b$ for the other, which are equal, as they ought. By a similar process, the substitution of the other value of x will give each member $=0$. Now, to obtain the interpretation, we have merely to take $n=1$, in equation (1.). We thus obtain $\sqrt{(x^2+a^2)}=x+b$: whence, by squaring and contracting, we get $a^2=2bx+b^2$; and consequently $x=\frac{a^2-b^2}{2b}$, the value required.* It may be remarked, that in this case the problem becomes the simplest possible, being merely this: Given one leg of a right-angled triangle, and the difference between the hypotenuse and the other; to find the latter.

The case in which b is negative, and which implies that n times the hypotenuse (n being evidently a proper fraction) is less than the required leg, presents no difficulty; and the expression for x will be found to be the same as the one obtained above, except that the first term of the numerator would be $-b$ and not b. If b were $=0$, we should have x simply $=\frac{na\sqrt{(1-n^2)}}{1-n^2}=\frac{na}{\sqrt{(1-n^2)}}$.†

Exam. 10.

Here, equation (2.) is obtained by squaring the members of (1.), and transposing 2. We then clear (2.) of fractions; and (4.) and (5.) are obtained by transposition, &c.‡

$$\sqrt{\frac{x-2}{x+2}}+\sqrt{\frac{x+2}{x-2}}=4 \ldots\ldots (1.)$$

$$\frac{x-2}{x+2}+\frac{x+2}{x-2}=14 \ldots\ldots (2.)$$

$$2x^2+8=14x^2-56 \ldots\ldots (3.)$$

$$12x^2=64 \ldots\ldots (4.)$$

$$x=\pm\frac{4}{\sqrt{3}}=\pm\frac{4}{3}\sqrt{3} \ldots\ldots (5.)$$

---

* This value may also be obtained from $\frac{b-n\sqrt{\{b^2-(n^2-1)a^2\}}}{n^2-1}$, by multiplying its numerator and denominator by $b+n\sqrt{\{b^2-(n^2-1)a^2\}}$; as the numerator and denominator of the result can each be divided by n^2-1: and by taking $n=1$ in what is obtained, the same value as above will be found for x.

† This is a solution of the question, in which one leg of a right-angled triangle and the ratio of the hypotenuse and the other are given, to find the latter leg.

‡ The following solution of this example, taken from Kelland's *Alge-*

154. The methods that have been established for the resolution of quadratic equations enable us to resolve trinomial equations of any degree whatever, which contain only two powers of the unknown quantity, provided the index of the one power be

bra, pp. 122. and 123., shows the disadvantage of introducing the subsidiary quantity y in this instance, though such substitutions are in many cases beneficial.

" Given $\sqrt{\frac{x-2}{x+2}}+\sqrt{\frac{x+2}{x-2}}=4$, to find x.

" Here we observe, that if $\sqrt{\frac{x-2}{x+2}}$ be called y, the equation becomes

$y+\frac{1}{y}=4$.

" Let us first solve this equation: it is,

$$y^2-4y=-1,$$

$$\therefore\quad y^2-4y+4=3,$$

$$y=2\pm\sqrt{3},$$

$$\text{or,}\quad \sqrt{\frac{x-2}{x+2}}=2\pm\sqrt{3}=c \text{ suppose};$$

$$\therefore\quad x-2=c^2(x+2) \text{ by squaring,}$$

$$x=\frac{2c^2+2}{1-c^2}=2\cdot\frac{1+c^2}{1-c^2}$$

$$=2\cdot\frac{1+(4\pm4\sqrt{3}+3)}{1-(4\pm4\sqrt{3}+3)}$$

$$=2\cdot\frac{8\pm4\sqrt{3}}{-6\mp4\sqrt{3}}=4\cdot\frac{2\pm\sqrt{3}}{-3\mp2\sqrt{3}};$$

the signification of the double sign being, that, if the upper one be + the lower is −, and *vice versa.*

" This value of x may be simplified in the following manner: —

$$-3\mp2\sqrt{3}=-(3\pm2\sqrt{3})$$

$$=-\sqrt{3}(\sqrt{3}\pm2)$$

$$\therefore\quad x=\frac{4}{-\sqrt{3}}\cdot\frac{2\pm\sqrt{3}}{\sqrt{3}\pm2}.$$

If the upper sign be taken, we get

$$x=-\frac{4}{\sqrt{3}}:$$

if the lower,

$$x=-\frac{4}{\sqrt{3}}\,\frac{2-\sqrt{3}}{\sqrt{3}-2}:$$

$$=\frac{4}{\sqrt{3}}:$$

therefore the values of x are

$$\pm\frac{4}{\sqrt{3}}."$$

exactly double of the index of the other. To show this, let us take the general equation (1.), where n is any number, positive or negative, whole or fractional. Then, assuming $x^n=y$, as equation (2.), we get equation (3.) by squaring: and equation (4.) is found from (1.) by substituting y^2 for x^{2n}, and y for x^n. Equation (5.) is derived from (4.) by means of § 151.; and (6.) is the same as (5.), except, that instead of y, its equal x^n is written. Then, in the last place, (7.) is obtained from (6.) by taking the nth root. It thus appears from (6.) that we obtain the lower of the two powers of x by means of § 151.; and thence the value of x itself by extracting the nth root.

$$ax^{2n}+bx^n=c \quad \ldots\ldots\ldots\ldots (1.)$$
$$x^n=y \quad \ldots\ldots\ldots\ldots (2.)$$
$$x^{2n}=y^2 \quad \ldots\ldots\ldots\ldots (3.)$$
$$ay^2+by=c \quad \ldots\ldots\ldots\ldots (4.)$$
$$y=\frac{-b\pm\sqrt{(b^2+4ac)}}{2a} \quad \ldots\ldots (5.)$$
$$x^n=\frac{-b\pm\sqrt{(b^2+4ac)}}{2a} \quad \ldots\ldots (6.)$$
$$x=\left\{\frac{-b\pm\sqrt{(b^2+4ac)}}{2a}\right\}^{\frac{1}{n}} \ldots (7.)$$

It would be shown in a similar way, that § 152. may be employed, when the second coefficient is even.

Exam. 11.

$$3x^4-2x^2=225 \quad \ldots\ldots\ldots (1.)$$
$$x^2=\frac{1\pm\sqrt{(1+675)}}{3} \quad \ldots\ldots (2.)$$
$$x^2=\frac{1\pm26}{3} \quad \ldots\ldots\ldots (3.)$$
$$x^2=9, \text{ and } x^2=-\tfrac{25}{3} \quad \ldots\ldots (4.)$$
$$x=\pm3, \text{ and } x=\pm\sqrt{-\tfrac{25}{3}} \,.\, (5.)$$

Here (2.) is found from (1.) by means of § 152. Then (3.) is found from (2.) by contracting the radical, and extracting its root. We finally get $x=3$, $x=-3$, $x=\sqrt{-\frac{25}{3}}$, and $x=-\sqrt{-\frac{25}{3}}$; the first and second values being real, and the others imaginary.

In this example we get (2.) by means of § 151. From the first value of $x^{\frac{2}{3}}$ in (3.) we get (4.) by cubing; and thence we obtain (5.) by extracting the square root. We might also get expressions for x from the second value of $x^{\frac{2}{3}}$ in (3.); but they would be imaginary.

Exam. 12.

$$5x^{\frac{4}{3}}+7x^{\frac{2}{3}}=108 \quad \ldots\ldots\ldots (1.)$$
$$x^{\frac{2}{3}}=\frac{-7\pm\sqrt{(49+2160)}}{10} \,.\, (2.)$$
$$x^{\frac{2}{3}}=4, \text{ and } x^{\frac{2}{3}}=-\tfrac{27}{5} \ldots (3.)$$
$$x^2=64 \quad \ldots\ldots\ldots\ldots (4.)$$
$$x=\pm8 \quad \ldots\ldots\ldots\ldots (5.)$$

In Example 13. equation (2.) is got from (1.) by means of § 47.; and (3.) from this by § 152. Equation (5.) is the same as (4.) with the indicated operations performed, and the mode of writing x^{-1} changed; and the values in (6.) are the reciprocals of those in (5.). We have thus an instance in which an equation is resolved without clearing it of fractions.

Exam. 13.

$$\frac{48}{x^2}+\frac{32}{x}=11 \ldots\ldots\ldots\ldots (1.)$$

$$48x^{-2}+32x^{-1}=11 \ \ldots\ldots (2.)$$

$$x^{-1}=\frac{-16\pm\sqrt{(256+528)}}{48} . (3.)$$

$$x^{-1}=\frac{-16\pm 28}{48} \ldots\ldots\ldots (4.)$$

$$\frac{1}{x}=\tfrac{1}{4}, \text{ and } \frac{1}{x}=-\tfrac{11}{12} \ldots\ldots (5.)$$

$$x=4, \text{ and } x=-\tfrac{12}{11} \ \ldots\ldots (6.)$$

Exam. 14. Given the sum of four equidifferent numbers = 24, and their continual product = 945; to find them.

Here (§ 132.) the sum of the means as well as that of the extremes will be 12, the half of 24. Putting, therefore, $2x$ to denote the common difference, we shall have (§ 52.) the means $= 6-x$, and $6+x$; and (§ 130.) the extremes will be $6-3x$, and $6+3x$. Now (§ 57.) the product of the means is $36-x^2$, and that of the extremes $36-9x^2$. The product of these two results is then found, and placed equal to 945; and thus we get equation (1.). From this (2.) is obtained by changing the order of the terms, and, for a contraction, dividing by 9. Then (3.) is found by transposition, and (4.) by § 152. We finally get $x=\sqrt{39}$, and $x=1$. Taking the latter of these, we find the four numbers ($12-3x$, $12-x$, &c.) to be 3, 5, 7, and 9, which satisfy the conditions of the question. If we take the first value of x, we find the four numbers to be $6-3\sqrt{39}$, $6-\sqrt{39}$, $6+\sqrt{39}$, and $6+3\sqrt{39}$; and these will also be found to answer. The first and second of them, however, are negative; and, therefore, in the ordinary arithmetical meaning, it is only 3, 5, 7, and 9, that are to be regarded as the answers to the question. In this problem, it is unnecessary to take $-\sqrt{39}$ and -1, the negative values of x, as they would give the same terms in a reversed order. Thus, if -1 had been taken, the terms of the series would have been 9, 7, 5, and 3.

$$1296-360x^2+9x^4=945 \ \ldots (1.)$$

$$x^4-40x^2+144=105 \ldots\ldots (2.)$$

$$x^4-40x^2=-39 \ \ldots\ldots\ldots (3.)$$

$$x^2=20\pm\sqrt{(400-39)} \ \ldots\ldots (4.)$$

$$x^2=39, \text{ and } x^2=1 \ \ldots\ldots\ldots (5.)$$

$$x=\sqrt{39}, \text{ and } x=1 \ \ldots\ldots (6.)$$

155. The principle established in the last § sometimes gives an easy and elegant method of resolving equations containing radicals, without previously removing the radicals by involution according to § 110. This application of the principle will be understood from the following examples.

Exam. 15. Let it be required to resolve the equation, $x-\sqrt{(3x-2)}=14$, without previous involution.

$$x-(3x-2)^{\frac{1}{2}}=14 \dots\dots\dots\dots (1.)$$
$$3x-3(3x-2)^{\frac{1}{2}}=42 \dots\dots\dots\dots (2.)$$
$$3x-2-3(3x-2)^{\frac{1}{2}}=40 \dots\dots\dots (3.)$$
$$(3x-2)^{\frac{1}{2}}=\frac{3\pm\sqrt{(9+160)}}{2} \dots\dots (4.)$$
$$(3x-2)^{\frac{1}{2}}=8, \text{ and } (3x-2)^{\frac{1}{2}}=-5 \dots (5.)$$
$$3x-2=64, \text{ and } 3x-2=25 \dots\dots (6.)$$
$$x=22, \text{ and } x=9 \dots\dots\dots\dots (7.)$$

In solving this, because the radical contains the term $3x$, we multiply by 3 in equation (1.) to get (2.). Then, by subtracting 2, we get equation (3.), in which $3x-2$ and its square root both occur in connexion with 1 (the coefficient of $3x-2$), -3, and 40, which are known quantities. In the next place, considering $(3x-2)^{\frac{1}{2}}$ as the quantity to be determined, we find (§ 151.) the values of that quantity to be 8 and -5, as in equation (5.). Thence, by squaring, we get (6.); and, from the expressions thus obtained, we find $x=22$, and $x=9$ in (7.). The former of these is the value of x, which satisfies the question in the terms in which it was proposed; and, therefore, it is naturally to be regarded as the answer. The other value 9, however, will also answer it, if $(3x-2)^{\frac{1}{2}}$ be taken negative; as we should then have $9-(-5)$, or $9+5$, which is equal to 14, as it ought.

Exam. 16. Resolve the equation, $x^3-26=9\sqrt{(x^3-4)}$, without previous involution.

$$x^3-26=9(x^3-4)^{\frac{1}{2}} \dots\dots\dots\dots (1.)$$
$$x^3-4-9(x^3-4)^{\frac{1}{2}}=22 \dots\dots\dots (2.)$$
$$(x^3-4)^{\frac{1}{2}}=\frac{9\pm\sqrt{(81+88)}}{2} \dots\dots (3.)$$
$$(x^3-4)^{\frac{1}{2}}=11, \text{ and } (x^3-4)^{\frac{1}{2}}=-2 \dots (4.)$$
$$x=5, \text{ and } x=2 \dots\dots\dots\dots (5.)$$

Here we get equation (2.) from (1.) by adding 22 to both members, and by transposing $9(x^3-4)^{\frac{1}{2}}$. Then the coefficient of x^3-4, regarded as a single term, being 1, and that of its square root being -9, we get (3) by § 151. From this the two parts of (4.) are obtained by using first the upper, and then the lower sign of the radical: and from these, by squaring, transposing, and extracting the cube root, we get the values of x in (5.). The first of these satisfies the given equation, if $\sqrt{(x^3-4)}$ be taken positive; the other, if it be taken negative. The former, therefore, is the answer to the question understood in the plain arithmetical sense.

It will be seen from these examples, that the first object, in such cases, is to modify the given equation so that its second member may be a known quantity, and that its first may consist of a quantity, simple or compound, with a known coefficient, together with another which is the square root of the same quantity having also a known coefficient. When such a modification cannot be made, this method is inadmissible. The process will be simplified to the student, when commencing, if a single character be put for the radical. Thus, in Exam. 15., we might put $(3x-2)^{\frac{1}{2}}=y$; and then equation (3.) would become $y^2-3y=40$. In like manner, in Exam. 16., we might put $(x^3-4)^{\frac{1}{2}}=y$, and equation (2.) would become $y^2-9y=22$.

156. When two or more simultaneous equations are given, one or more of the unknown quantities must generally * be eliminated, till an equation is obtained which contains only one unknown quantity: and if this equation be of the second degree, it will be resolved by one or other of the modes that have been explained. The elimination will be effected by means of some of the methods explained in Chapter VIII.; and most usually by the second or third method.

Exam. 17. Find the values of x and y in the equations, $3x-4y=3$, and $x^2-y^2=16$.

Here, equation (3.) is derived from (1.) by transposition and division; and (4.) from (3.) by squaring. Then, by substituting the value of y^2, thus obtained, in (2.), we get (5.), which does

$$3x-4y=3 \dots\dots\dots\dots (1.)$$

$$x^2-y^2=16 \dots\dots\dots\dots (2.)$$

$$y=\frac{3x-3}{4} \dots\dots\dots\dots (3.)$$

$$y^2=\frac{9x^2-18x+9}{16} \dots\dots\dots (4.)$$

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* In some particular instances, we may advantageously proceed otherwise. Thus, if the equations, $x-y=a$, and $(x+y)^{2n}+b(x+y)^n=c$, were proposed, we might find $x+y$ by means of § 154.; and then the values of x and y would be found by means of § 52.

not contain y. The rest of the operation proceeds in the usual way, and presents no difficulty. In addition to the positive values, $x=5$, and $y=3$, we have the two negative ones, $x=-7\frac{4}{7}$, and $y=-6\frac{3}{7}$.

$$x^2-\frac{9x^2-18x+9}{16}=16 \quad \ldots (5.)$$

$$16x^2-9x^2+18x-9=256 \;. \; (6.)$$

$$7x^2+18x=265 \quad \ldots\ldots\ldots (7.)$$

$$x=\frac{-9\pm\sqrt{(81+1855)}}{7} \quad \ldots (8.)$$

$$x=5\text{, and } x=-7\tfrac{4}{7} \ldots\ldots (9.)$$

$$y=3\text{, and } y=-6\tfrac{3}{7} \ldots\ldots (10.)$$

As a slight variation of the process, we might have found the value of y^2 from (2.), and have put it equal to its value in (4.): or we might have commenced by eliminating x.

Exercises. Find the values of x in the following equations.

1. $18+x^2=180-x^2$ *Ans.* $x=\pm 9$.
2. $x-\frac{44}{x}=\frac{198}{x}-x$. *Ans.* $x=\pm 11$.
3. $\sqrt{(a^2+x^2)}=bx$. *Ans.* $x=\pm\frac{a}{\sqrt{(b^2-1)}}$.
4. $\frac{x+a}{x-a}+\frac{x-a}{x+a}=b$. *Ans.* $x=\pm a\sqrt{\frac{b+2}{b-2}}$.
5. $\frac{a}{x+\sqrt{(2a^2-x^2)}}+\frac{a}{x-\sqrt{(2a^2-x^2)}}=\frac{x}{a}$. *Ans.* $x=\pm a\sqrt{2}$.
6. $\frac{x^2+ax+b}{x^2+bx+c}=\frac{a}{b}$. *Ans.* $x=\pm\sqrt{\frac{b^2-ac}{a-b}}$.
7. $x^2+6x=7$. *Ans.* $x=1$, and $x=-7$.
8. $x^2-6x=7$. *Ans.* $x=-1$, and $x=7$.
9. $\frac{1}{3}x^2-\frac{1}{2}x=9$. *Ans.* $x=6$, and $x=-4\frac{1}{2}$.
10. $20x-x^2=51$. *Ans.* $x=17$, and $x=3$.
11. $x^2-x=\frac{1}{4}(3x^2-7x+70.)$ *Ans.* $x=7$, and $x=-10$.
12. $2x-7=10-x+\sqrt{(65-x^2)}$. *Ans.* $x=7$, and $x=3\frac{1}{5}$.
13. $x-\sqrt{(2x^2-9x+1)}=1$. *Ans.* $x=7$, and $x=0$.
14. $x-\sqrt{(ax^2-bx+c^2)}=c$. *Ans.* $x=\frac{b-2c}{a-1}$, and $x=0$.
15. $4x^2+17x=x+468$. *Ans.* $x=9$, and $x=-13$.
16. $3x+5\sqrt{x}=22$. *Ans.* $x=4$, and $x=13\frac{4}{9}$.
17. $2x^2+2x+2=24-5x$. *Ans.* $x=2$, and $x=-5\frac{1}{2}$.
18. $x^2-6x=6x+28$. *Ans.* $x=14$, and $x=-2$.

19. $8+x-\frac{26}{8-x}=1$. *Ans.* $x=6$, and $x=-5$.

20. $x+\frac{4}{x}=\frac{6}{x}+1$. *Ans.* $x=2$, and $x=-1$.

21. $\frac{12}{x}+\frac{10}{x-1}=9$. *Ans.* $x=3$, and $x=\frac{4}{9}$.

22. $x+ax^2=\frac{a}{n}$. *Ans.* $x=\frac{-n\pm\sqrt{(n^2+4na^2)}}{2na}$.*

23. $x+x^{-1}=a$. *Ans.* $x=\frac{1}{2}\{a\pm\sqrt{(a^2-4)}\}$.

24. $x^n+x^{-n}=a$. *Ans.* $x=\{\frac{1}{2}[a\pm\sqrt{(a^2-4)}]\}^{\frac{1}{n}}$.

25. $(x-3)^2+(x+4)^2=(x+5)^2$. *Ans.* $x=8$, and $x=0$.

26. $(x-2)^2+(x+5)^2=(x+6)^2$. *Ans.* $x=7$, and $x=-1$.

27. $\sqrt[3]{(x+5)^2}+3\sqrt[3]{(x+5)}=28$. *Ans.* $x=59$, and $x=-348$.

28. $x-y=6$, and $x^2+y^2=50$. *Ans.* $x=7$, and $y=1$; or $x=-1$, and $y=-7$.

29. $x+2y=5$, and $x^2+2y^2=11$. *Ans.* $x=3$, and $y=1$; or $x=\frac{1}{3}$, and $y=\frac{7}{3}$.

30. $x+ny=a$, and $x^2+ny^2=b$.

Ans. $x=\frac{a\pm\sqrt{\{(n^2+n)b-na^2\}}}{n+1}$, and

$$y=\frac{na\mp\sqrt{\{(n^2+n)b-na^2\}}}{n^2+n}.$$

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* If n be very large in comparison of a, so that, the coefficient of x being unity, the absolute term $\frac{a}{n}$ may be very small in comparison of the coefficient of x^2, it will appear, from this result, that the positive value of x will be a small fraction. To show this, divide the numerator and denominator of that value of x by n; then $x=\frac{-1+\sqrt{(1+4a^2n^{-1})}}{2a}$; an expression in which, because of n being very large in comparison of a, the radical part is very little greater than 1. Hence the numerator is very small; and, the denominator being $2a$, this value of x must be a very small quantity.

Thus, suppose a were $=1$ and $n=100$, the positive value of x would be found to be less than 0·01. It would appear, in fact, from § 126., that, if n be *very great* in comparison of a, the positive value of x will be very nearly $\frac{a}{n}$, the terms in the square root of $1+a^2n^{-1}$, after the first and second, being extremely small in value.

31. $\frac{x}{y}+\frac{y}{x}=a$, and $xy=b$. *Ans.* $x=\frac{1}{2}\{\sqrt{(ab+2b)}+\sqrt{(ab-2b)}\}$,
$y=\frac{1}{2}\{\sqrt{(ab+2b)}-\sqrt{(ab-2b)}\}$.

32. $xy=36$, and $y+2\sqrt{(x-y)}=x-3$. *Ans.* $x=12$, $y=3$; or $x=\frac{1}{2}(\sqrt{145}+1)$, and $y=\frac{1}{2}(\sqrt{145}-1)$.

33. $x^2+\sqrt{(x^2-x-6)}=x+48$.
Ans. $x=7$ or -6; or $x=\frac{1}{2}(1\pm\sqrt{221})$.

34. $\frac{49}{x^2+2xy+y^2}+\frac{28}{x+y}=5$, and $\frac{7}{x^2-2xy+y^2}-\frac{5}{x-y}=2$.
Ans. $x=4$, and $y=3$; or $x=-\frac{49}{20}$, and $y=\frac{21}{20}$.

*Miscellaneous Exercises in Quadratic Equations.**

1. Given the sum of two numbers $=8$ $(=s_1)$, and the sum of their squares $=50$ $(=s_2)$; to find the numbers.

Ans. 7 and 1; general expression, $\frac{1}{2}\{s_1\pm\sqrt{(2s_2-s_1^2)}\}$.

2. Given the sum of two numbers $=7$ $(=s_1)$, and the sum of their cubes $=133$ $(=s_3)$; to find the numbers.

Ans. 5 and 2; general expression, $\frac{1}{2}\left\{s_1\pm\sqrt{\left(\frac{4s_3}{3s_1}-\frac{s_1^2}{3}\right)}\right\}$.

3. Given the difference of two numbers $=2$ $(=d_1)$, and the difference of their cubes $=218$ $(=d_3)$; to find the numbers.

Ans. 7 and 5, or -5 and -7; general expressions,

$$\frac{1}{2}\left\{d_1\pm\sqrt{\left(\frac{4d_3}{3d_1}-\frac{d_1^2}{3}\right)}\right\}, \text{ and } \frac{1}{2}\left\{-d_1\pm\sqrt{\left(\frac{4d_3}{3d_1}-\frac{d_1^2}{3}\right)}\right\}.$$

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* In solving these exercises, it may perhaps be better for the beginner merely to use the particular numbers, and thus to get the numerical answers; but when he has had more practice, he ought to employ the characters *a*, *b*, *p*, &c., so as to get the general solutions, and thence to find the numerical answers by substituting for those letters their particular values. As usual, the positive results alone would be recognised as the solutions to the questions in their literal meanings; but the learner should find the others, and endeavour to discover their interpretations. Some of them may be better solved by other means than by employing the rules that have been established for resolving quadratics. Such in particular is Exercise 8. The student should compare Exercises 9. and 10. with 4. and 5. He ought also to compare Exercises 13. and 14., and likewise 15. and 16. In Exercise 17. he will find the time during which the first person travelled to be 11 days or −60 days. Let him consider the meaning of the latter result.

4. Given the sum of two numbers $= 22$ $(=s_1)$, and their product $= 117$ $(=p)$; to find the numbers.

Ans. 13 and 9; general expression, $\frac{1}{2}\{s_1 \pm \sqrt{(s_1^2-4p)}\}$.

5. Given the difference of two numbers $= 5$ $(=d_1)$, and their product $= 36$ $(=p)$; to find the numbers.

Ans. 9 and 4, or -4 and -9; general expressions,
$\frac{1}{2}\{d_1 \pm \sqrt{(d_1^2+4p)}\}$, and $\frac{1}{2}\{-d_1^2 \pm \sqrt{(d_1^2+4p)}.\}$

6. Given the product of two numbers $= 18$ $(=p)$, and the sum of their squares $= 85$ $(=s_2)$; to find the numbers.

Ans. 9 and 2, or -9 and -2;
general expressions, $\frac{1}{2}\{\sqrt{(s_2+2p)} \mp \sqrt{(s_2-2p)}\}$,
and $\frac{1}{2}\{\sqrt{(s_2+2p)} \pm \sqrt{(s_2-2p)}\}$.

7. Given the product of two numbers $= 40$ $(=p)$, and the difference of their squares $= 39$ $(=d_2)$; to find the numbers.

Ans. 8 and 5, or -8 and -5;
general values, $x = \pm\sqrt{\{\frac{1}{2}d_2 \pm \frac{1}{2}\sqrt{(d_2^2+4p^2)}\}}$,
and $y = \pm\sqrt{\{-\frac{1}{2}d_2 \pm \frac{1}{2}\sqrt{(d_2^2+4p^2)}\}}$.

8. Find two numbers, such, that the cube of their sum may exceed the sum of their cubes by 60 $(=a)$, and the difference of their cubes may exceed the cube of their difference by 36 $(=b)$.

Ans. 4 and 1; general expressions,

$$\sqrt[3]{\frac{(a+b)^2}{6(a-b)}} \text{ and } \sqrt[3]{\frac{(a-b)^2}{6(a+b)}}.$$

9. Given the product of two numbers $= 6$, and the sum of their cubes $= 35$; to find the numbers. *Ans.* 2 and 3.

10. Given the product of two numbers $= 20$, and the difference of their cubes $= 61$; to find the numbers. *Ans.* 5 and 4.

11. The product of two numbers is 240 $(=a)$, and, if one of them be increased by 4 $(=b)$, and the other be diminished by 3 $(=c)$, the product of the results is still the same. Required the numbers.

Ans. 16 and 15, or -20 and -12; general expressions,

$$\frac{-bc \pm \sqrt{(b^2c^2+4abc)}}{2c}, \text{ and } \frac{bc \pm \sqrt{(b^2c^2+4abc)}}{2b}.$$

12. Find four numbers in arithmetical progression, such that their continual product may be 105 $(=a)$, and that the product of the means may exceed that of the extremes by 8 $(=b)$.

Ans. 1, 3, 5, 7, and -1, -3, -5, -7; or,
in general, $x-3y$, $x-y$, $x+y$, and $x+3y$,
where $x = \pm\frac{1}{4}\sqrt{\{10b \pm 8\sqrt{(b^2+4a)}\}}$, and $y = \pm\sqrt{\frac{1}{8}b}$.

13. A person purchased a certain number of oxen for one hundred and twenty pounds, and he found that, had he got three more for the same money, they would each have cost him two pounds less. How many did he purchase? *Ans.* 12.

14. A person purchased a certain number of oxen for one hundred and twenty pounds, and he found that if he had got three fewer for the same money, they would each have cost him two pounds more than they did. How many did he purchase? *Ans.* 15.

15. A company at a tavern had a reckoning of seven pounds four shillings to pay; but, two persons being exempted from paying, the rest had each to give one shilling more than they otherwise would. How many persons were in the company? *Ans.* 18.

16. A company at a tavern had a reckoning of seven pounds four shillings to pay; but, two of them having paid who had been intended to be exempted, the rest had each to contribute one shilling less than they otherwise would. How many persons were there, exclusive of the two? *Ans.* 16.

17. Suppose two towns, A and B, to be 300 miles asunder, and that one traveller sets out from A towards B, travelling the first day 10 miles, the second 11, the third 12, and so on; while, two days after, another starts from B for A, and travels every day 15 miles: where will they meet? *Ans.* 165 miles from A.

18. Find four numbers in geometrical progression, such that the sum of the extremes may be 35, and that of the means 30.
Ans. 8, 12, 18, and 27.

19. Two couriers start at the same time from two towns, A and B, and travel each towards the other town; after some days they meet, when it is found that one of them has travelled 84 miles more than the other, and that by continuing to travel each at the same rate as he had done before, the one will finish the journey in 9 days, and the other in 16. Required the distance of the places, and the rates at which the couriers travelled.

Ans. Distance 588 miles; rates 28 miles and 21 miles daily;
or distance 12 miles; rates 4 miles and -3 miles daily.*

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* Of the various modes in which this question may be solved, the following is an outline of perhaps the simplest and most elegant. Let t be the time that elapsed from starting till the couriers were together, and $x+42$ and $x-42$, the spaces travelled by each in that time; then,

$$x+42 : x-42 :: t : 9, \text{ and } x-42 : x+42 :: t : 16.$$

Take the products of the corresponding terms of these two analogies; then $t^2=144$, and therefore $t=\pm 12$. Use each of these values of t in either of the foregoing analogies, and the values of x will be found by

157. The methods of resolving quadratic equations having been fully exemplified and illustrated, we may now proceed to consider the relations of their roots, and to deduce some results of an interesting kind connected with the theory of such equations.

Equations of the second degree may have four distinct forms. Thus, if a, b, and c are numbers which are in themselves positive, we shall have

1. $ax^2+bx=c$; where $x=\dfrac{-b\pm\sqrt{(b^2+4ac)}}{2a}$:

2. $ax^2-bx=c$; where $x=\dfrac{b\pm\sqrt{(b^2+4ac)}}{2a}$:

3. $ax^2-bx=-c$; where $x=\dfrac{b\pm\sqrt{(b^2-4ac)}}{2a}$: and

4. $ax^2+bx=-c$; where $x=\dfrac{-b\pm\sqrt{(b^2-4ac)}}{2a}$.

Now, in the first and second of these the radical part is real; and it is greater than b, since the square root of b^2+4ac is greater than that of b^2, that is, than b. In the third and fourth, the radical is real, when b^2 is not less than $4ac$; but, if $4ac$ be greater than b^2, it is imaginary, as b^2-4ac is then negative. Farther, also, in these two forms, when the radical is real, it is less than b; as the square root of b^2-4ac is less than the square root of b^2, that is, b. Hence, therefore, when the absolute term is positive, as in the first and second forms, the roots are always real; but, when it is negative, as in the third and fourth forms, the roots are imaginary, when $4ac$ is greater that b^2; otherwise they are real.

Now, in the first and second forms, the radical, being greater than b, may be represented by $b+d$. Then, the numerators of the roots in the first form will be $-b+b+d$ and $-b-b-d$, or, by contraction, d and $-2b-d$; one of which is positive and the other negative: and, in absolute magnitude, the positive one is less than the negative. If these be divided by $2a$, the quotients will be the roots; and in these the same relations will still exist: that is, one of the roots is positive and the other negative; and, in

equalling the products of the extremes and means. The second answers correspond to the question in which the couriers must have travelled in the same direction, and in which the first would require nine days to travel back to B, and the second sixteen to travel to A; the sum of the spaces travelled, before being together, being 84 miles.

absolute magnitude, the positive is the less. In the second form, the numerators of the roots would be found in a similar manner to be $2b+d$ and $-d$: and, by reasoning as before, it would be found, that, in that form, one root is positive, and the other negative; and that, in absolute magnitude, the positive is the greater.

In the third and fourth forms, we have seen that, when the roots are real, the radical is less than b; it may be represented, therefore, by $b-d$, where, it is plain, d lies between the limits 0 and b. Then, in the third form, the numerators of the roots will be $b+b-d$ and $b-(b-d)$, or, by contraction, $2b-d$, and d, which are both positive; and, if these be divided by the denominator $2a$, the quotients, that is, the roots of the equation, will be positive. In the fourth form, the numerators of the roots would be found to be $-d$, and $-2b+d$, which are both negative, d being less than b, and consequently than $2b$; and, since by dividing these by $2a$ we should get the roots, it follows that, in this form, the roots, when real, are both negative.

Lastly, in the third and fourth forms, if $b^2=4ac$, the radical vanishes, and the roots are equal; being each positive in the third form, and each negative in the fourth.

158. The principal results obtained above may be briefly stated in the following manner. When the absolute term c is positive, the roots are real; and one of them is positive and the other negative. In this case also, when b, the coefficient of the second term, is positive, the positive root is less in absolute magnitude than the negative one: but, if b be negative, the positive root is the greater. When the absolute term is negative, the roots are real, if $4ac$ be not greater than b^2: otherwise they are imaginary. When they are real, if b, the coefficient of the second term, be negative, they are both positive; but, if b be positive, they are both negative. It may be remarked, in the last place, that the roots are either both real or both imaginary, a single imaginary root not occurring in any case.

The relations thus established will be illustrated by means of the following equations, which the student will do well to resolve, and to consider in connexion with those relations.

1. $12x^2+5x=25.$
2. $12x^2-5x=25.$
3. $6x^2-29x=-28.$
4. $6x^2+29x=-28.$
5. $3x^2-5x=-4.$
6. $3x^2+5x=-4.$

159. If we divide all the terms of the equation, $ax^2+bx=c$, by a, and put, for brevity and simplicity, $\frac{b}{a}=2p$ and $\frac{c}{a}=q$, we

shall have $x^2+2px=q$, or, by transposition, $x^2+2px-q=0$; where p and q may be any numbers, whole or fractional, positive or negative, according to the data of any particular problem. By resolving this by means of § 152., we get $x=-p\pm\sqrt{(p^2+q)}$, an expression which contains the two values of x. Writing these separately, and putting x_1 to denote the one having the upper sign, and x_2 the other, we have, $x_1=-p+\sqrt{(p^2+q)}$, and $x_2=-p-\sqrt{(p^2+q)}$. By adding these, the radicals disappear, and we get $x_1+x_2=-2p$: whence it appears, that, *when the coefficient of the highest power in a quadratic equation is* 1, *the sum of its roots is equal to the second term with its sign changed.* Thus, in Exam. 4., the sum of the roots 8 and 3, with its sign changed, is equal to -11, the coefficient of the second term in equation (4.).

160. By taking the products of x_1 and x_2, and of their values in § 159., we get (§ 57.) $x_1x_2=p^2-(p^2+q)=-q$; and, therefore, *the product of the two roots of a quadratic equation having unity as the coefficient of its first term, is equal to the absolute term with its sign changed, or to that term, if it be transposed to the first member.*

Thus, in Exam. 4., the product of the roots, 8 and 3, is 24, the same as the absolute term in equation (4), with its sign changed.

161. If the values of x in § 159. be attached to that letter with their signs changed, we shall have $x+p-\sqrt{(p^2+q)}=0$, and $x+p+\sqrt{(p^2+q)}=0$. Taking the product of these, we get by § 57., $(x+p)^2-(p^2+q)=0$; from which, by actually squaring $x+p$, and subtracting p^2+q from the result, we get, $x^2+2px-q=0$, which is the original equation. Hence it appears, that, *if the roots be attached to x with their signs changed, and if the results be multiplied into one another, the product, put equal to* 0, *will be the original equation;* and, conversely, *if it be required to find what factors of the first degree will produce any proposed quantity of the second degree,* we have merely to put that quantity equal to nothing; to find the roots of the equation so obtained, and to attach them separately to x with their signs changed; as the expressions thus found will be the factors.

Thus, if the expression $x^2-2x-15$ be proposed, by putting it equal to 0, and resolving the equation thus found, we get $x=5$ and $x=-3$; and therefore the factors are $x-5$ and $x+3$, the product of which will be found to be the proposed expression; and thus the correctness of the process is verified.

162. If the quantity of which the factors are to be found be of

the form ax^2+bx-c, where a is different from unity, the factors are obtained in the simplest manner, by putting that quantity equal to 0, and finding the roots by § 151 or § 152. Then, by what we have seen, the product of the factors obtained by changing the signs of the roots, and attaching them separately to x, would give the expression that would arise from dividing ax^2+bx+c by a; it is therefore necessary to multiply one of the factors by a, or to multiply one of them by one factor of that quantity, and the other by its remaining factor. We shall find, in fact, that this object will be accomplished by freeing both the factors so found of fractions, if both be fractional, or one of them if only that one be fractional.

Thus, to find the factors of $6x^2-17x+12$, we put this expression $=0$; and, by resolving the equation so obtained, according to § 151., we get $x=\frac{3}{2}$, and $x=\frac{4}{3}$. Hence, the factors of $x^2-\frac{17x}{6}+2$ are $x-\frac{3}{2}$, and $x-\frac{4}{3}$; but that quantity being only one sixth of the proposed one, the product of these factors must be increased six fold; and this will evidently be accomplished, if the one factor be doubled, and the other trebled; so that the required factors are $2x-3$ and $3x-4$, which will be found to succeed. In like manner, we should find the simple factors of $2x^2+5x-3$ to be $x+3$ and $2x-1$; and those of $12x^2+31x+20$ to be $4x+5$ and $3x+4$.

163. The method of establishing the important principles contained in §§ 159, 160, 161, and 162, may perhaps be regarded as too synthetic: and, on this account, the following mode of treating the subject, which also affords a very simple and elegant investigation of the method of resolving quadratic equations, will perhaps be considered preferable.

Let us assume, as in § 159., $x^2+2px=q$: then, by adding p^2, we get equation (2.). Equation (3.) is the same as (2.), k^2 being put, for brevity, to denote p^2+q; whence we have k equal to the radical, $\sqrt{(p^2+q)}$. By transposing k^2 we find (4.); and thence, by § 57., we get (5.). Now there are two ways, and only two, in which equation (5.), and consequently those from

$$x^2+2px=q \;\ldots\ldots\ldots\ldots\; (1.)$$

$$x^2+2px+p^2=p^2+q \;\ldots\ldots\; (2.)$$

$$(x+p)^2=k^2 \;\ldots\ldots\ldots\ldots\; (3.)$$

$$(x+p)^2-k^2=0 \;\ldots\ldots\ldots\; (4.)$$

$$(x+p-k)(x+p+k)=0 \;.\; (5.)$$

$$x+p-k=0 \;\ldots\ldots\ldots\ldots\; (6.)$$

$$x+p+k=0 \;\ldots\ldots\ldots\ldots\; (7.)$$

$$x=-p+k \;\ldots\ldots\ldots\ldots\; (8.)$$

$$x=-p-k \;\ldots\ldots\ldots\ldots\; (9.)$$

which it is derived, can be satisfied, that is, can have the first member $= 0$. One of these modes is to take the first factor equal to 0, and the other to take the second equal to the same: and thus we get equations (6.) and (7.). That there can be no other mode of satisfying equation (5.) is plain; since, if both its factors be different from 0, their product cannot be equal to 0. Equations (8.) and (9.) are derived from (6.) and (7.) by transposition; and, by writing, in place of k, its value according to the assumption in equation (3.), we get the same values for x that we should have obtained by means of § 152.; and thus we obtain the means of resolving quadratic equations in a manner very different from that formerly employed. We see also, that, in such equations, the unknown quantity can have two values, and only two.

164. By examining the last operation, it will be seen that (4.) is the same as $x^2+2px-q$, k^2 being the same as p^2+q; and, therefore, the subtracting of it from $(x+p)^2$, or $x^2+2px+p^2$, will give for remainder $x^2+2px-q$. The process, in fact, was such as, without any change of value, virtually to reduce $x^2+2px-q$ to the form (4.). Now, according to (5.), the first member of (4.), and consequently what is identical with it, $x^2+2px-q$, is found to be the product of the two factors in (6.) and (7.); and these, according to (8.) and (9.), are the same as x with its two values attached to it, with their signs changed: and we thus see that the trinomial, $x^2+2px-q$, is produced by the multiplication of two factors which are found in the manner above described.

165. In the equation, $y^2=ax^2+bx+c$, it is of importance, in certain cases, to determine what values of x render y real, and what others render it imaginary; or, which is the same, to find what values of x will make ax^2+bx+c positive, and what negative. To effect this, divide the second member by a; and (§ 161.) find $x-x_1$ and $x-x_2$, the factors of the quotient. Then, by multiplying the product of these by a, we shall plainly have $y^2=a(x-x_1)(x-x_2)$. Let us now consider the case in which x_1 and x_2 are real. If, then, $x=x_1$, or $x=x_2$, we shall have $y^2=0$; but, for every other value of x, y^2 will have a value either positive or negative. Of other values of x there can plainly be three classes, and no more; as any of its values must either be intermediate between x_1 and x_2, or must be greater than either, or less than either. In the first of these cases, one of the factors, $x-x_1$ and $x-x_2$ will be positive and the other negative, x being greater than one of the quantities, x_1 and x_2, and less

than the other. The product, therefore, of these factors will be negative; and, accordingly, y^2 will be positive if a be negative, and negative if a be positive. In each of the other cases, however, the sign of y^2 will be the same as that of a: for if x be greater than each of the two quantities x_1 and x_2, the factors $x-x_1$ and $x-x_2$ will both be positive; and therefore their product will be positive: but, if x be less than each of the two, x_1 and x_2, the factors will be both negative; and therefore their product will also be positive. Since, therefore, y^2 is equal to that product multiplied by a, it follows that y^2 will be positive when a is positive, and negative when a is negative.

166. The conclusions now arrived at may be recapitulated in the following terms. If there be an equation $y^2=ax^2+bx+c$, and if the values of x in the equation, $ax^2+bx+c=0$, be determined, and be called x_1 and x_2; then, when these are real, if x be taken equal to either of these, y^2 will be equal to 0; secondly, when x has any value intermediate between x_1 and x_2, the sign of y^2 is the opposite of that of a; but, when x has any other value, y^2 has the same sign as a.

Thus, for example, if $y^2=x^2-4$, we have $a=1$, $b=0$, and $c=-4$; and we get $x_1=2$, and $x_2=-2$. Then, a being positive, any value of x between 2 and -2, will render y^2 negative; while values beyond these limits will make it positive. Thus, if $x=\pm 1$, $y^2=-3$; if $x=0$, $y^2=-4$, &c.: while, if $x=\pm 3$, $y^2=5$, &c. For values of x, therefore, between 2 and -2, y is imaginary; but for all others it is real.

If, for another example, $y^2=4-x^2$, we have now $a=-1$; and $x_1=2$, and $x_2=-2$, as before: and, as a is negative, values of x, such as 1, -1, &c., between 2 and -2, will render y^2 positive; while values beyond these limits will render it negative.

For a third example, let $y^2=3x^2-5x-12$. Here a is positive, and we find $x_1=3$, and $x_2=-\frac{4}{3}$: every value of x, therefore, lying between 3 and $-\frac{4}{3}$, will give y imaginary; while others will make it real. Thus, if $x=2$, $y=\pm\sqrt{-10}$; but if $x=-2$, $y=\pm\sqrt{10}$.

If, again, $y^2=2x^2-12x+18$, we get x_1 and x_2 each equal to 3. Then, as there can be no values of x intermediate between these, and as the coefficient of x^2 is positive, y^2 will be positive, whatever value, except 3, is assumed for x. It may be remarked, that, in this example, we have $y^2=2(x-3)^2$; and, as $(x-3)^2$ must always be positive, except when $x=3$, its product by 2 must have the sign of 2. In this case, we have $y=\pm(x-3)\sqrt{2}$;

and there is always a similar relation, when x_1 and x_2 are equal to one another.

167. Let us now consider the case in which x_1 and x_2 are imaginary; that is (§ 158.), when $4ac$ is negative, and is greater in absolute value than b^2. In this case, let $4ac=b^2+d$, where d is necessarily positive. Then $c=\frac{b^2}{4a}+\frac{d}{4a}$; and, therefore $y^2=ax^2+bx+c$ becomes

$$y^2=ax^2+bx+\frac{b^2}{4a}+\frac{d}{4a}, \text{ or } y^2=a\left(x+\frac{b}{2a}\right)^2+\frac{d}{4a},$$

the latter value being evidently equivalent to the former. Now, the two parts of the latter form of the value of y^2 have evidently the same sign as a; one of them being a square, which is necessarily positive, multiplied by a, and the other a quantity, which, as we have seen, is also positive, divided by $4a$. It follows, therefore, that *if the values of x found by putting $ax^2+bx+c=0$ be imaginary, every value of y^2 will have the same sign as the coefficient of x^2, whatever value may be assumed for x.*

As an example, let $y^2=x^2-2x+5$. By putting the second member of this $=0$, we get $x=1\pm2\sqrt{-1}$, which is imaginary. Then, since 1, the coefficient of x^2 is positive, every value given to x must make y^2 positive. Thus, if $x=0$, $y^2=5$; if $x=1$, $y^2=4$, and if $x=-2$, $y^2=13$.

As another example, let $y^2=-2x^2+2x-1$. Then, by putting this $=0$, we get $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{-1}$: and this being imaginary, and the coefficient of x^2 negative, every value of x will give y^2 negative; and therefore y can have no real value whatever.

168. It may be stated in the last place, regarding the trinomial, ax^2+bx+c, that it will be an exact square, if $b^2=4ac$. This is evident from § 151.; for we saw there, that when a trinomial is a square, its middle term is twice the product of the square roots of the first and third. Hence, that ax^2+bx+c may be a square, we must have $bx=2a^{\frac{1}{2}}c^{\frac{1}{2}}x$; and from this, by dividing by x, and squaring, we get $b^2=4ac$.

169. Since $(\sqrt{x}+\sqrt{y})^2=x+y+2\sqrt{xy}$, it follows, conversely, that $\sqrt{(x+y+2\sqrt{xy})}=\sqrt{x}+\sqrt{y}$. In some cases, however, and especially when x and y are numbers expressed in the common notation, the traces of those quantities, x and y, are lost by their incorporation; and we get an expression of the form, $a+\sqrt{b}$, where a is equal to the sum of the rational parts, x and y,

and $\sqrt{b}$ equal to the irrational part, $2\sqrt{xy}$ or $\sqrt{4xy}$. Thus, for instance,

$$(\sqrt{3}+\sqrt{2})^2=3+2+2\sqrt{3\times 2}=5+2\sqrt{6}=5+\sqrt{24}:$$
$$\text{and } \{\sqrt{(x+y)}-\sqrt{(x-y)}\}^2=2x-2\sqrt{(x^2-y^2)}.$$

If we wish, therefore, to extract the square root of $a+\sqrt{b}$, we have merely to divide a into two parts, x and y, such that twice the product of their square roots may be $\sqrt{b}$, and consequently $4xy=b$: and thus we have equations (1.) and (2.). Equation (3.) is obtained from (1.) by squaring; (4.) from (2.) and (3.) by subtracting; and (5.) from (4.) by extracting the square root. Then, by taking half the sum and half the difference of (1.) and (5.), we get (6.) and (7.). Hence, the required root being $\sqrt{x}+\sqrt{y}$, we shall have

$$x+y=a \ldots\ldots\ldots \text{(1.)}$$
$$4xy=b \ldots\ldots\ldots \text{(2.)}$$
$$x^2+2xy+y^2=a^2 \ldots \text{(3.)}$$
$$x^2-2xy+y^2=a^2-b \;.\; \text{(4.)}$$
$$x-y=\sqrt{(a^2-b)} \ldots \text{(5.)}$$
$$x=\tfrac{1}{2}\{a+\sqrt{(a^2-b)}\} \;.\; \text{(6.)}$$
$$y=\tfrac{1}{2}\{a-\sqrt{(a^2-b)}\} \;.\; \text{(7.)}$$

$$\sqrt{(a+\sqrt{b})}=\sqrt{\tfrac{1}{2}\{a+\sqrt{(a^2-b)}\}}+\sqrt{\tfrac{1}{2}\{a-\sqrt{(a^2-b)}\}}.$$

If the proposed quantity were $a-\sqrt{b}$, the required root would evidently be found by subtracting the square root of the second member of (7.) from that of the second member of (6.), instead of adding them.

These formulas are used with advantage, when a^2-b is an exact square. When this is not so, they are of scarcely any use, on account of their being complicated.

As an example, let it be required to extract the square root of $5+2\sqrt{6}$.

Here we have $a=5$, $\sqrt{b}=2\sqrt{6}$; and therefore $b=24$. Then $x=\frac{1}{2}\{5+\sqrt{(25-24)}\}=3$, and $y=\frac{1}{2}\{5-\sqrt{(25-24)}\}=2$. Hence the root is $\sqrt{3}+\sqrt{2}$.

If again the square root of $11-4\sqrt{7}$ is to be extracted, we have $a=11$, and $b=112$, the square of $-4\sqrt{7}$; and by equations (6.) and (7.), we get $x=7$, and $y=4$; and therefore the required root, $\sqrt{x}-\sqrt{y}$, is $\sqrt{7}-2$.

Lastly, let it be required to extract the square root of $2v+2\sqrt{(v^2-z^2)}$.

Here, $a=2v$, and $b=4v^2-4z^2$; and therefore $a^2-b=4z^2$; and $\sqrt{(a^2-b)}=2z$. Hence, by equations (6.) and (7.), we have $x=v+z$, and $y=v-z$: and therefore the required root is $\sqrt{(v+z)}+\sqrt{(v-z)}$.

Exercises.

Find the square roots of the following quantities.

Exercises.	Answers.	Exercises.	Answers.
1. $8\pm\sqrt{60}$	$\sqrt{5}\pm\sqrt{3}$.	4. $76\pm32\sqrt{3}$	$8\pm2\sqrt{3}$.
2. $6\pm2\sqrt{5}$	$\sqrt{5}\pm1$.	5. $39\pm6\sqrt{42}$	$\sqrt{21}\pm3\sqrt{2}$.
3. $49\pm12\sqrt{13}$	$6\pm\sqrt{13}$.	6. $52\pm30\sqrt{3}$	$5\pm3\sqrt{3}$.

7. $p^2+2\sqrt{(p^2q^2-q^4)}$. *Ans.* $\sqrt{(p^2-q^2)}+q$.

8. $4a^2-(4a-2b)\sqrt{(4ab-b^2)}$ *Ans.* $2a-b-\sqrt{(4ab-b^2)}$.

9. $2a^2+2b^2\pm2\sqrt{(a^4+a^2b^2+b^4)}$.

Ans. $\sqrt{(a^2+ab+b^2)}\pm\sqrt{(a^2-ab+b^2)}$.

CHAPTER X.

GENERAL THEORY OF EQUATIONS, AND RESOLUTION OF EQUATIONS OF THE HIGHER ORDERS.

170. WHEN an equation contains only one unknown quantity, x, it is either of the form,

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0\ \ldots\ldots\ (a);$$

where n is a whole positive number, and $p_1, p_2, \ldots\ldots, p_n$, coefficients, either whole or fractional, positive or negative, and independent of x: or, if it be not of that form, it may be reduced to it by involution, transposition, division, or other operations already explained. Thus, the equation, $x^3-4x+5=0$, is comprehended in (a), n being $=3$, $p_1=0$, $p_2=-4$, and p_3, or in the present case, $p_n=5$. If, again, $2x^2-x+\sqrt{(3x-4)}=0$, we get, by transposing the radical, squaring, again transposing, and dividing by 4, $x^4-x^3+\frac{1}{4}x^2-\frac{3}{4}x+1=0$; which is also comprehended in equation (a), n being $=4$, $p_1=-1$, $p_2=\frac{1}{4}$, $p_3=-\frac{3}{4}$, and p_4, or here $p_n=1$: and so in all other cases. Hence, in the following Chapter all the equations treated of will be of the same form as equation (a). Instead, however, of writing such an equation at full length, we may often, for brevity, write it $fx=0$; f being the first letter of the term *function*, and fx may be read *function of x*. The term *function*, which is much used by modern writers on mathematics, *denotes, that the quantity which it expresses depends on the one of which it is a function,* Thus, x^3 is a function of x, since, being the third power of x, its value necessarily depends on that of x. In like manner, also, ax^{-n} and $\sqrt{(a^2-ax+x^2)}$ are functions of x. Again, if $x = -\frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2+b)}$, x is a function of the *two* quantities, a and b; and we may write the equation thus: $x=f(a, b)$. In this instance x is either of the roots of the quadratic equation, $x^2+ax=b$: whence it appears, that the roots of a quadratic equation are functions of the coefficients of its terms [*]: and the same is plainly the case with regard to the roots of all equations whatever; as the values of the roots must obviously depend on those of the coefficients. It will thus be seen that fx may denote an infinite number of different functions of x. The function, however, which it will be employed to express in this Chapter, is the first member of equation (a).

171. The following is an important proposition in the theory of equations. *If a be a root of the equation, $fx=0$, fx is divisible without remainder by $x-a$.* To prove this, let fx be divided by $x-a$; and let the quotient be denoted by $f'x$, and the remainder, if there be any, by r. Then, by the nature of division, $fx=(x-a)f'x+r$. Now, a being a root of the equation, $fx=0$, the substitution of a for x will render $fx=0$; a root of an equation being a number which, if it be substituted for the unknown quantity, will make the two members equal. The factor $x-a$, also, will become $a-a$, or 0: and therefore the foregoing expression will become $0=0\times f'x+r$, or simply, $0=r$; that is, the remainder is 0.

172. If it be taken for granted, that every equation has at least one root; that is, that there is a value of the unknown quantity, such that, if it be substituted for that quantity, the members of the equation will be rendered equal to one another [†]; the

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[*] When the equation is written thus, $x^2+ax-b=0$, $-b$ may be regarded as the coefficient of x^0.

[†] While no one hesitates in admitting that every equation has a root, it is difficult to prove it. Besides other attempts, a demonstration has been given by Cauchy, *Cours d'Analyse*, pp. 331—339., which appears to be satisfactory; but it is too difficult, and too little elementary, to be given here. If the equation arise from the expression of the conditions of a problem in the language of algebra, with a view to find an unknown quantity, the existence of that quantity, that is, of a root of the equation, is implied in the very nature of the inquiry; and even should the question be impossible, in consequence of the incompatibility of some of the data, this impossibility would be pointed out by the circumstance, that the values which would be obtained for the unknown quantity, would be

property just established leads to the following proposition, which is one of the most important in the theory of equations. *The first member of equation* (*a*) *is the product of n factors of the first degree, and of the forms* $x-a$, $x-b$, ... , $x-k$, $x-l$; *where a, b, c, ... , k, l are the roots of the equation.* To prove this, suppose a to be a root of the equation,

$$fx=0, \text{ or } x^n+p_1x^{n-1}+\ \ldots\ +p_n=0,$$

and let the first member of that equation be supposed to be divided by $x-a$. Then, by § 171., there will be no remainder; and the quotient will evidently be of the form

$$x^{n-1}+p'_1x^{n-2}+\ \ldots\ +p'_{n-1};$$

so that, conversely, $fx=(x-a)(x^{n-1}+p'_1x^{n-2}+\ \ldots\ +p'_{n-1})$. Suppose, again, b to be a root of the equation,

$$x^{n-1}+p'_1x^{n-2}+\ \ldots\ +p'_{n-1}=0.$$

Then, by § 171., the first member of this is divisible by $x-b$; and the quotient will be of the form,

$$x^{n-2}+p''_1x^{n-3}+\ \ldots\ +p''_{n-2}:$$

and consequently,

$$x^{n-1}+\ \ldots\ +p'_{n-1}=(x-b)(x^{n-2}+\ \ldots\ +p''_{n-2}).$$

By substituting this in the value just found for fx, we get

$$fx=(x-a)(x-b)(x^{n-2}+\ \ldots\ +p''_{n-2}).$$

By proceeding in a similar manner with c, a root of the equation $x^{n-2}+\ \ldots\ +p''_{n-2}=0$, we should introduce a new factor, $x-c$: and as, by the division by each new factor, the index of the highest power of x in the quotient is less by unity than the corresponding index of the dividend, the successive divisions may be continued till, after $n-2$ such divisions, a quotient is obtained which is only of the second degree: and this (§ 161.) will be the product of two factors of the first degree, such as $x-k$ and $x-l$. Hence, therefore, we shall have

$$fx=(x-a)(x-b)\ \ldots\ (x-k)(x-l);$$

where the number of the factors is plainly n. Now this will become 0; that is, the equation will be satisfied, if any of the factors, $x-a$, &c., be equal to nothing, and in no other case. The first of these factors will become nothing, when $x=a$, as the

imaginary. These imaginary values, too, if substituted for the unknown quantity, would satisfy the equation; and they are, therefore, to be regarded as roots of it. For much on this subject, see Young's *Theory of Equations*, second edition, chap. iii.

factor will then become $a-a$ or 0, instead of $x-a$. The second, in like manner, will become nothing, when $x=b$; the third, when $x=c$; and so on. Hence the roots of the equation are $a, b, c, \ldots, l$; since, if any one of these be substituted for x, one of the factors of fx, and therefore fx itself, will vanish. Thus, if $x=b$, we should have $(b-a)(b-b)\ldots\ldots(b-k)(b-l)$, which is equal to nothing, since the second factor $b-b=0$. It is plain also, that the equation can have no other roots besides these; as no value for x, except $a, b, c, \ldots\ldots, l$, could make any of the factors become nothing; and a product can become nothing only when one or more of its factors vanish. *An equation,* therefore, *can have as many roots as there are units in the index of the highest power of the unknown quantity, and no more.*

173. From § 172. we see, conversely, that if the roots of an equation be given, the equation will be found by attaching them severally, with their signs changed, to x, taking the continual product of the quantities thus obtained, and putting it $=0$. Thus, if the roots be $a, b, c, \ldots\ldots$, the equation will be

$$(x-a)(x-b)(x-c)\ldots=0:$$

and if, as a particular example, the roots be 2, -3, and 4, the equation will be $(x-2)(x+3)(x-4)=0$; or, by actual multiplication, $x^3-3x^2-10x+24=0$; an equation which will be satisfied by taking $x=2$, $x=-3$, or $x=4$.

174. It also follows from § 172., that if we know one root a of an equation, $fx=0$, and if we divide fx by $x-a$, and put the quotient $=0$, we shall have an equation of a lower degree, the roots of which will be the remaining roots of $fx=0$. Thus, if we know that one root of the equation, $x^3+x^2-8x-12=0$, is -2, we get, by dividing the first member by $x+2$, and putting the quotient $=0$, $x^2-x-6=0$; the roots of which quadratic, 3 and -2, are the remaining roots of the proposed equation.

It is plain also, that, by attaching to x each of two or more known roots, with their signs changed, by taking the product of the results, and by dividing fx by it, the quotient, when put $=0$, will be an equation, the roots of which will be the remaining roots of $fx=0$.

175. When an equation, $fx=0$, which has its coefficients real, has imaginary roots, they occur *in pairs* of the form $\alpha+\alpha'\sqrt{-1}$ and $\alpha-\alpha'\sqrt{-1}$; the two differing only in the sign of the imaginary part. In proving this, we see, in the first place, that $fx=0$ cannot have a *single* imaginary root: for, if it

could have $\alpha+\alpha'\sqrt{-1}$ as a root, without any other imaginary one, then (§ 173.) we should have

$$fx=(x-a)(x-b)\ldots.(x-\alpha-\alpha'\sqrt{-1});$$

and, if the actual multiplication were performed, the product of the part, $x-\alpha$ by $(x-a)(x-b)\ldots.$ would evidently be real, but that of $-\alpha'\sqrt{-1}$ by the same would have all its coefficients imaginary. The entire product, therefore, could not be fx, which, by hypothesis, has only real coefficients.

Let us now examine, whether there can be *two* imaginary roots, $\alpha+\alpha'\sqrt{-1}$ and $\beta+\beta'\sqrt{-1}$; and if so, what must be the relations of the quantities α, α', β, and β'. If these roots be admissible, the product of the factors, $x-\alpha-\alpha'\sqrt{-1}$ and $x-\beta-\beta'\sqrt{-1}$, must be real. Now, that product is

$$(x-\alpha)(x-\beta)-(x-\alpha)\beta'\sqrt{-1}-(x-\beta)\alpha'\sqrt{-1}-\alpha'\beta'.$$

Of these four terms, the first and last are real: and therefore the entire product will be real, if the two remaining terms can be made to disappear. These terms may be written under the form, $-x(\alpha'+\beta')\sqrt{-1}+(\alpha\beta'+\alpha'\beta)\sqrt{-1}$; and this expression will vanish, if $\alpha'+\beta'=0$, and $\alpha\beta'+\alpha'\beta=0$. The first of these conditions gives $\beta'=-\alpha'$: and by substituting this in the second, and resolving, we get $\beta=\alpha$. Hence $fx=0$ may have two imaginary roots, the second becoming $\alpha-\alpha'\sqrt{-1}$, and the first remaining $\alpha+\alpha'\sqrt{-1}$, as it was assumed: and, by similar reasoning, it would appear, that, if there be other imaginary roots, there must be two, or some other even number of them; and that each pair must have the relation that has been established.*

It may be remarked, that the product of the two factors $x-\alpha-\alpha'\sqrt{-1}$ and $x-\alpha+\alpha'\sqrt{-1}$, is $(x-\alpha)^2+\alpha'^2$; which, as it is the sum of two squares, must (§ 118.) be positive: but (§ 173.) it must be equal to nothing; and hence the incongruity which gives origin to the imaginary roots.

As an example, let it be required to find the equation whose roots are 3, $2+2\sqrt{-1}$, and $2-2\sqrt{-1}$. Here, the two imaginary factors are $x-2-2\sqrt{-1}$ and $x-2+2\sqrt{-1}$; the product of which is x^2-4x+8. Multiplying this by $x-3$, and putting the product $=0$, we get $x^3-7x^2+20x-24=0$, the required equation.

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* Roots thus related are often called *conjugate roots.*

Exercises. Find, in terms of x, the equations whose roots are as follows.

1. $3, -2.$ *Ans.* $x^2-x-6=0.$
2. $2+3\sqrt{-1},\ 2-3\sqrt{-1}.$ *Ans.* $x^2-4x+13=0.$
3. $1, 2, 3.$ *Ans.* $x^3-6x^2+11x-6=0.$
4. $2, -4, 5, -6.$ *Ans.* $x^4+3x^3-36x^2-68x+240=0.$
5. $1, -1, \sqrt{-1}, -\sqrt{-1}.$ *Ans.* $x^4-1=0.$
6. $1, 1, 1, -1, -2.$ *Ans.* $x^5-4x^3+2x^2+3x-2=0.$

176. If, as in the annexed process, we perform the actual multiplication of the factors, $x-a$, $x-b$, $x-c$, and $x-d$, so as to get, first, (1.) the equation which has a and b as roots; then (2.) the one which has for roots, a, b, c; and, thirdly, the one marked (3.) whose roots are a, b, c, and d, we shall be able to discover from the results some important relations, which exist between the roots of equations and their coefficients.

$$\begin{array}{l}
x-a \\
x-b \\
\hline
\end{array}$$

$$\left.\begin{array}{r|l}
x^2-a & x+ab=0 \\
-b &
\end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\ (1.)$$

$$\begin{array}{l}
x-c \\
\hline
\end{array}$$

$$\left.\begin{array}{r|r|l}
x^3-a & x^2+ab & x-abc=0. \\
-b & +ac & \\
-c & +bc &
\end{array}\right\}\cdots\cdots\cdots\ (2.)$$

$$\begin{array}{l}
x-d \\
\hline
\end{array}$$

$$\left.\begin{array}{r|r|r|l}
x^4-a & x^3+ab & x^2-abc & x+abcd=0 \\
-b & +ac & -abd & \\
-c & +bc & -acd & \\
-d & +ad & -bcd & \\
 & +bd & & \\
 & +cd & &
\end{array}\right\}\cdots\ (3.)$$

By examining the three products, (1.), (2.), and (3.), we find that in each, the coefficient of the second term is the sum of all the roots, and the last term the product of them all, with their signs changed in both cases: and, by considering the mode in which each product is derived from the one before it, we shall find that this principle holds universally. Thus, for instance, the coefficient of the second term of (2.) is, by the nature of multiplication, composed of the coefficient of the corresponding term of (1.), together with the next root c, with its sign changed;

and to this coefficient, $-d$ is added to get the corresponding coefficient in (3.). The last terms also, in (1.), (2.), and (3.), are respectively the products of $-a$ and $-b$; of $-a$, $-b$, and $-c$; and of $-a$, $-b$, $-c$, and $-d$: and these relations must plainly hold, whatever may be the number of the roots.

It will be readily seen also, that the coefficient of each of the third terms is the sum of all the products that can be obtained by combining the roots by pairs; the fourth, with its sign changed, the sum of all the products that can be formed by taking three of the roots as the factors of each; and so on.*

177. From the last § it follows, that, if the second term of an equation be wanting, its roots must be partly positive and partly negative, and must balance one another. Thus, the equation $x^3-7x+6=0$, may be written $x^3\pm0x^2-7x+6=0$. Then, the coefficient of the second term being nothing, the sum of the roots must be nothing. The roots, in fact, are 1, 2, and -3; the algebraic sum of which is nothing. It follows also from the same §, that if the last term be wanting, that is, be equal to nothing, one or more of the roots must be equal to nothing; as the product of the roots could not otherwise be nothing. Thus, one root of the equation, $x^3-2ax^2-px=0$, is 0; and this, being substituted for x, satisfies the equation, as it ought. Then, dividing by x, which (§ 172.) must be a factor of the first member, we get $x^2-2ax-p=0$; the roots of which, $a\pm\sqrt{(a^2+p)}$, are the remaining roots of the proposed equation. In like manner, the roots of x^3-ax^2 are 0, 0, and a.

178. By examining the product found in § 176., we shall see, that, if we change the signs of the terms occupying the second, fourth, and other even places, we shall get an equation having its roots the same as those of the proposed equation with the opposite signs: as the signs of those terms will be changed without

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* The conclusions arrived at above might be established in a stricter manner according to the method employed in § 116. Thus, if the properties be true in an equation of the nth degree, it might be shown by multiplying by a factor $x-p$, that they will hold also in an equation of the next higher order. Now, since, as we have seen, they hold in an equation of the fourth degree, they must also hold in one of the fifth; and therefore also in one of the sixth; and so on.

The student will find it easy to prove the following additional property: *if the coefficient of the last term but one be divided by the last term, the quotient, with its sign changed, is equal to the sum of the reciprocals of the roots.*

changing the absolute magnitudes of the terms themselves, if the signs of all the quantities, $a, b, \ldots, l$, be changed; while the other terms, being each the product of an even number of factors, will not be affected by such a change. Thus, in the equation $x^2+5x-14=0$ the roots are 2 and -7; while those of $x^2-5x-14=0$ are -2 and 7: and in the equation $x^3-13x-12=0$, or $x^3\pm 0x^2-13x-12=0$, the roots are 4, -3, and -1; while those of $x^3-13x+12=0$ are -4, 3, and 1.[*]

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[*] The following rule, which was given by Descartes, and which is sometimes useful, is easily established: *In any equation the number of positive roots cannot exceed the number of changes of the signs of its successive terms, and the number of its negative roots cannot exceed the number of permanences among the same signs.* Thus, in the equation, $x^4+3x^3-36x^2-68x+240=0$ (Exer. 4. p. 170.), the signs are $+\,+\,-\,-\,+$. In these there are two changes of signs, one between the second and third, and one between the fourth and fifth; and therefore, according to the rule, there cannot be more than two positive roots. There are also two permanences of signs, one between the first and second, and one between the third and fourth; and therefore, if the rule be true, there cannot be more than two negative roots. These conclusions are correct, as the roots are 2, 5, -4, and -6.

The rule is easily shown to be true by multiplying a polynomial containing the successive powers of x, with several variations of signs, first by $x-a$, a factor corresponding to a positive root, a; and then by $x+a$, corresponding to a negative root, $-a$; and it will be seen, that the former process must always introduce at least one additional variation of the signs; while the latter, introducing no additional variation, but producing one additional term, will add at least one to the number of permanences. In doing this, it is sufficient to employ merely the signs, as in the margin. In these processes, when the quantities to be added have opposite signs, the double sign $\pm$ is used as what is to be prefixed to their sum; since, without knowing whether the positive or negative term is the greater, we cannot know whether the corresponding term in the product is positive or negative. Now, it is easy to see, that, in the first process, the double or ambiguous sign must occur as often as there are permanences in the multiplicand, and in the other as often as there are variations; so that it will appear on a little reflection, that in the first operation the number of permanences, and in the second the number of variations, cannot be increased; and as each product contains one term more than the multiplicand, it follows, that in the first the number of variations, and in the second the number of permanences, must be increased by at

$$\begin{array}{cccccccccc}
+ & - & + & + & + & - & - & + & - & \\
+ & - & & & & & & & & \\
\hline
+ & - & + & + & + & - & - & + & - & \\
 & - & + & - & - & - & + & + & - & + \\
\hline
+ & - & + & \pm & \pm & - & \pm & + & - & +
\end{array}$$

$$\begin{array}{cccccccccc}
+ & - & + & + & + & - & - & + & - & \\
+ & + & & & & & & & & \\
\hline
+ & - & + & + & + & - & - & + & - & \\
 & + & - & + & + & + & - & - & + & - \\
\hline
+ & \pm & \pm & + & + & \pm & - & \pm & \pm & -
\end{array}$$

179. Equations may often be transformed into others, which may be more easily resolved, or in other respects more easily managed, than the equations themselves. One of the most common of these transformations is that which is technically called *the taking away of the second term*, that is, the deriving of an equation wanting the second term from the given one. The method of effecting this will be understood from the annexed work, where the first line contains as much of equation (*a*), § 170., as is necessary.

$$x^n+p_1x^{n-1}+\text{\&c.} \quad \ldots\ldots\ldots \quad (a)$$

$$x=x'+y \quad \ldots\ldots\ldots \quad (1.)$$

$$x^n=x'^n+nx'^{n-1}y+\text{\&c.} \quad \ldots\ldots\ldots \quad (2.)$$

$$x^{n-1}=x'^{n-1}+\text{\&c.} \quad \ldots\ldots\ldots \quad (3.)$$

$$x^n+p_1x^{n-1}\ldots.=x'^n+nx'^{n-1}y+\text{\&c.}+p_1x'^{n-1}+\text{\&c.} \quad (4.)$$

$$ny+p_1=0 \quad \ldots\ldots\ldots \quad (5.)$$

$$y=-\frac{p_1}{n} \quad \ldots\ldots\ldots \quad (6.)$$

In equation (1.) x is assumed $=x'+y$, where y is the quantity to be determined. Equations (2.) and (3.) are obtained from (1.) by means of § 116.; and in (3.) the first term alone is sufficient in the present case. Equation (4.) is derived from (*a*) by substituting in it the values of x^n and x^{n-1}, found in equations (2.) and (3.). Now, in the second member of this, it is plain, that the second term, that which contains x'^{n-1}, will disappear, if its multiplier, $ny+p_1$, be $=0$; a condition which is expressed in equation (5.): and we get (6.) from (5.) by resolving it for y. Hence, therefore, to obtain the required equation, we have merely to attach to a new letter x', an nth part of the coefficient p_1 with its sign changed, and to substitute the result for x in the given equation.*

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least one; and this being so for every root that is introduced, the truth of the rule is evident from § 176. The subject will be illustrated by the exercises in p. 170.

It would be easily shown, that the part of the rule which regards positive roots will hold in incomplete equations; while, with regard to negative roots, the equations must be completed by inserting, with ambiguous signs, and with zeros as coefficients, the powers that are wanting.

* By assuming as above, $x=x'+y$, an equation may be found wanting the third or any other term. As such transformations, however, are rather curious than useful, they are merely suggested for the consideration of the student; and he will find that, to take away the third term, it would be necessary to resolve an equation of the second degree; to take away the next, one of the third degree, and so on; and that, to take away

As an example, let it be required to change the equation, $x^4-4x^3+3x^2-20=0$, into another wanting the second term. Here, by dividing 4, the coefficient of the second term with its sign changed, by 4, the index of the highest power, we get 1; and, therefore, in equation (1.), we assume $x=x'+1$. Then, equations (2.), (3.), and (4.), are found from (1.) by taking the second, third, and fourth powers; and (5.) is got from (3.) by multiplying by -4, and (6.) from (2.) by multiplying by 3. Now, the first members of (4.), (5.), (6.), and (7.), are the terms of the first member of the given equation: and, therefore, by adding together their second members, and putting the sum equal to 0, we get (8.), which is the required equation, as it does not contain x'^3.

$$\begin{aligned}
x &= x'+1 \ldots\ldots\ldots\ldots\ldots\ldots & (1.)\\
x^2 &= x'^2+2x'+1 \ldots\ldots\ldots\ldots\ldots & (2.)\\
x^3 &= x'^3+3x'^2+3x'+1 \ldots\ldots\ldots & (3.)\\
x^4 &= x'^4+4x'^3+6x'^2+4x'+1 \ldots & (4.)\\
-4x^3 &= -4x'^3-12x'^2-12x'-4 \ldots & (5.)\\
3x^2 &= 3x'^2+6x'+3 \ldots & (6.)\\
-20 &= -20 \ldots & (7.)\\
& x'^4-3x'^2-2x'-20=0 \ldots\ldots & (8.)
\end{aligned}$$

Exercises. From the following equations derive others wanting the second terms.

7. $x^2+2ax-b=0.$ *Ans.* $x'^2-a^2-b=0.$
8. $x^3+6x^2+12x-56=0$ *Ans.* $x'^3-64=0.$
9. $x^3-2x^2-3=0.$ *Ans.* $x'^3-\frac{4}{3}x'-3\frac{16}{27}.$
10. $x^4-8x^3+5=0.$ *Ans.* $x'^4-24x'^2-64x'-43.$

180. The most important transformation of equations is that which is technically called *the increasing or diminishing of the roots of an equation by a given quantity;* that is, when there is an equation given, the finding of another in terms of a new unknown quantity, such that its roots may be greater or less, by an assigned quantity, than those of the given equation. The following examples will make the student acquainted with the method of effecting this transformation.

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the last term, it would be necessary to resolve an equation of the nth degree; in fact, virtually, to resolve the proposed equation. He will also find, if he choose to go through with the investigation, that the taking away of the second term will also destroy the third, if the coefficients of these terms be so related, that $(n-1)p_1^2=2np_2$. An instance of this occurs in Exercise 8.

Exam. 1. Let it be required to increase the roots of the equation $x^3-4x^2-2x+15=0$ by 2; that is to transform this equation into one in terms of x', x' being $=x+2$.

To effect this, let the first member be divided by $x+2$, which is done most easily by the method of detached coefficients, as in the margin. We thus find the quotient to be $x^2-6x+10$, with -5 as remainder. Then, by the nature of division, we shall have the first member, $fx=(x^2-6x+10)(x+2)-5$. Dividing again, as in the margin, $x^2-6x+10$ by $x+2$, we get as quotient $x-8$, with the remainder 26; and therefore, by the nature of division, we have $x^2-6x+10=(x-8)(x+2)+26$. By multiplying this by $x+2$, and substituting the product in the foregoing value of fx, we get $fx=(x-8)(x+2)^2+26(x+2)-5$. Lastly, dividing $x-8$ by $x+2$, as in the margin, we get 1, with -10 remaining; and therefore, by multiplying by $x+2$, we get $x-8=x+2-10$. Multiplying this by $(x+2)^2$, and substituting the product in the last value of fx, we obtain $(x+2)^3-10(x+2)^2+26(x+2)-5=0$; or $x'^3-10x'^2+26x'-5=0$, by writing x' instead of $x+2$. In this equation, the roots (the values of x') will evidently be each greater by 2 than those of the original equation (the values of x), since $x'=x+2$. In fact the roots of the given equation are 3, $\frac{1}{2}+\frac{1}{2}\sqrt{21}$, and $\frac{1}{2}-\frac{1}{2}\sqrt{21}$; while those of the other are 5, $2\frac{1}{2}+\frac{1}{2}\sqrt{21}$, and $2\frac{1}{2}-\frac{1}{2}\sqrt{21}$.

$$\begin{array}{rrrr|r} 1 & -4 & -2 & 15 & -2 \\ & -2 & 12 & -20 & \\ \hline & -6 & 10 & -5 & \end{array}$$

$$\begin{array}{rrr|r} 1 & -6 & 10 & -2 \\ & -2 & 16 & \\ \hline & -8 & 26. & \end{array}$$

$$\begin{array}{rr|r} 1 & -8 & -2 \\ & -2 & \\ \hline & -10 & \end{array}$$

By uniting the foregoing processes by means of the detached coefficients, the work will stand as in the margin. In this form each line is one step shorter than the preceding; and the coefficients of the required equation are 1, the first coefficient in the given one, and -10, 26, and -5, the last numbers in the several columns. We thus see, how, by means of the coefficients, the required equation is found by an easy, uniform, and continuous process.

$$\begin{array}{rrrr|r} 1 & -4 & -2 & 15 & -2 \\ & -2 & 12 & -20 & \\ \hline & -6 & 10 & -5 & \\ & -2 & 16 & & \\ \hline & -8 & 26 & & \\ & -2 & & & \\ \hline & -10 & & & \end{array}$$

Exam. 2. Let it be required to find an equation, having its roots each less by 3, than those of the equation, $x^4+3x^3-6x^2-3x+5=0$.

Here we are to divide the first member by $x-3$; the quotient by $x-3$; the next quotient by $x-3$; and the quotient so obtained, still by the same divisor. It will be seen from the annexed process by means of the coefficients, that the first quotient is $x^3+6x^2+12x+33$, with the remainder, 104; that the next quotient is $x^2+9x+39$, with the remainder, 150; that the next quotient and remainder are $x+12$ and 75; and that the fourth and last quotient is 1, with the remainder 15. Hence, by proceeding as in the last example, we shall have the following transformations of the value of fx:

$$\begin{array}{rrrrr|l}
1 & 3 & -6 & -3 & 5 & 3 \\
 & 3 & 18 & 36 & 99 & \\ \hline
 & 6 & 12 & 33 & 104 & \\
 & 3 & 27 & 117 & & \\ \hline
 & 9 & 39 & 150 & & \\
 & 3 & 36 & & & \\ \hline
 & 12 & 75 & & & \\
 & 3 & & & & \\ \hline
 & 15 & & & &
\end{array}$$

$$\begin{aligned}
fx &= x^4+3x^3-6x^2-3x+5=0 \\
&= (x^3+6x^2+12x+33)(x-3)+104 \\
&= \{(x^2+9x+39)(x-3)+150\}(x-3)+104 \\
&= (x^2+9x+39)(x-3)^2+150(x-3)+104 \\
&= \{(x+12)(x-3)+75\}(x-3)^2+150(x-3)+104 \\
&= (x+12)(x-3)^3+75(x-3)^2+150(x-3)+104 \\
&= (x-3+15)(x-3)^3+75(x-3)^2+150(x-3)+104 \\
&= (x-3)^4+15(x-3)^3+75(x-3)^2+150(x-3)+104.
\end{aligned}$$

By writing, therefore, x' instead of $x-3$, we get

$$fx=x'^4+15x'^3+75x'^2+150x'+104=0;$$

an equation in which the values of x' will severally be less than those of x by 3, because $x'=x-3$. A process of the same kind may evidently be employed with regard to any similar equation whatever; and thus we have the means of solving this important problem, in every case that can present itself.*

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* The following investigation, though not quite so easily followed out, is more general than the method given in the text; and it will therefore be preferred by the student, as soon as he shall be able readily to understand it.

Let $f_1x=x^n+p_1x^{n-1}+\ldots\ldots+p_n=0$; and let it be required to find an equation having its roots severally less by r (greater if r be negative) than those of f_1x. Divide f_1x, and what is equivalent to it, by $x-r$, and there will result

Exercises. In Exer. 11., diminish the roots by 2; in 12., by 4; and in 13., by 5; and in the next increase the roots by 1.

11. $x^2-4x+3=0$. *Ans.* $x'^2-1=0$.
12. $x^3-13x-12=0$. *Ans.* $x'^3+12x'^2+35x'=0$.

$\frac{f_1x}{x-r}=f_2x+\frac{R_1}{x-r}$; and consequently, $f_1x=f_2x(x-r)+R_1 \ldots\ldots (1.)$; where f_2x is an expression of the $(n-1)$th degree in terms of x, such as $x^{n-1}+p'_1x^{n-2}+$, &c., and R_1 a remainder independent of x. In a similar manner, by successive divisions by $x-r$, &c., we should get

$$f_2x=f_3x(x-r)+R_2 \ldots\ldots (2.);$$
$$f_3x=f_4x(x-r)+R_3 \ldots\ldots (3.);$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f_{n-1}x=f_nx(x-r)+R_{n-1} \ldots\ldots (n-1);$$

where f_3x, f_4x, ..., and f_nx, are expressions of the $(n-2)$th, $(n-3)$th, ..., and 1st orders; while the expressions, R_1, R_2, ..., and R_{n-1} are mere numbers. Now, since the coefficient of x^n in f_1x is unity, it follows from the nature of division, that the coefficients of the highest powers of x in the expressions, f_2x, f_3x, ..., and f_nx, are each unity. Hence, f_nx must be of the form $x+A$; and therefore we have

$$\frac{f_nx}{x-r}=1+\frac{A+r}{x-r}, \text{ or } \frac{f_nx}{x-r}=1+\frac{R_n}{x-r},$$

by putting $A+r=R_n$; whence $f_nx=x-r+R_n \ldots\ldots (n.)$

By substituting this value of f_nx in equation $(n-1)$, we obtain

$$f_{n-1}x=(x-r)^2+R_n(x-r)+R_{n-1}.$$

By substituting this, in like manner, in equation $(n-2)$, we should get

$$f_{n-2}x=(x-r)^3+R_n(x-r)^2+R_{n-1}(x-r)+R_{n-2}:$$

and, by similar substitutions, we should at length obtain

$$f_1x=(x-r)^n+R_n(x-r)^{n-1}+R_{n-1}(x-r)^{n-2}+\ \ldots\ +R_1=0 \ldots (a.)$$

Hence, by putting $x-r=x'$, we find for the required equation,

$$x'^n+R_nx'^{n-1}+R_{n-1}x'^{n-2}+\ \ldots\ +R_2x'+R_1=0 \ldots\ldots (b.)$$

To obtain f_2x, f_3x, &c., by a simpler and easier mode of division than that which is commonly employed, let us assume

$$f_2x=x^{n-1}+p'_1x^{n-2}+p'_2x^{n-3}+\ \ldots\ +p'_{n-1}:$$

Then, from (1.) we have $x^n+p_1x^{n-1}+p_2x^{n-2}+\ \ldots\ldots\ +p_n$

$$=(x^{n-1}+p'_1x^{n-2}+p'_2x^{n-3}+\ \ldots\ldots +p'_{n-1})(x-r)+R_1;$$

or by actual multiplication,

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ \ldots\ldots\ +p_{n-1}x+p_n=$$
$$\begin{array}{r|r|r|r} x^n+p'_1 & x^{n-1}+p'_2 & x^{n-2}+\ \ldots\ldots\ +p'_{n-1} & x+R_1 \\ -r & -rp'_1 & -rp'_{n-2} & -rp'_{n-1}. \end{array}$$

Now, by the nature of division, these two expressions are identical; and, therefore, we must have $p'_1-r=p_1$, or, by transposition, $p'_1=p_1+r$. In

13. $x^4-125x+4=0$. *An.* $x'^4+20x'^3+150x'^2+375x'+4=0$.

14. $x^3+3x^2-3=0$. *Ans.* $x'^3-3x'-1=0$.

181. If in equation (a), § 170, we take $x=x'q^{-1}$, so that $x'=qx$, we shall get $x'^nq^{-n}+\ldots\ldots+p_n=0$; whence, by multiplying by q^n, we have

$$x'^n+qp_1x'^{n-1}+q^2p_2x'^{n-2}+q^3p_3x'^{n-3}+\ldots\ldots+q^np_n=0.$$

We have thus the means of deriving, from a given equation, another having its roots in any ratio (that of q to 1) to those of the given equation. Thus, in the equation $x^3-3x^2+2x+5=0$, by taking $x=\frac{1}{10}x'$ we have $q=10$, and we get $x'^3-30x'^2+200x'+5000=0$; an equation which has each of its roots ten times the corresponding root of the given equation.

182. The formula in the last § is useful in enabling us to derive from an equation having its first coefficient $=1$, and having one or more of its other coefficients fractional, an equation having also its first coefficient $=1$, and its others integers. Thus, suppose the equation $x^3-\frac{5}{6}x^2-\frac{3}{4}x+1=0$, were given, we find (*Arithmetic*, p. 84.) the least common multiple of 6 and 4 to be 12; and, by taking q equal to this, we get

like manner, we must have $p'_2-rp'_1=p_2$; whence $p'_2=p_2+rp'_1$. In a similar way we should get

$$p'_3=p_3+rp'_2,\ \ldots\ldots,\ p'_{n-1}=p_{n-1}+rp'_{n-2},\text{ and }R_1=p_n+rp'_{n-1}.$$

Hence we have the following form of the division by $x-r$:

$$\begin{array}{cccccc|c} 1 & p_1 & p_2 & p_3\cdots\cdots & p_{n-1} & p_n & r \\ & \underline{r} & \underline{rp'_1} & \underline{rp'_2\cdots\cdots} & \underline{rp'_{n-2}} & \underline{rp'_{n-1}} & \\ & p'_1 & p'_2 & p'_3\cdots\cdots & p'_{n-1} & R_1. & \end{array}$$

Now this process is nothing else than the operation by means of the detached coefficients, as explained in § 50.; for the first line consists of $1, p_1, p_2, \ldots\ldots$, the coefficients in f_1x: also 1, the first coefficient is multiplied by r, and the product is placed under p_1; and we have seen that p'_1 is the sum of these two. We then multiply p'_1, so found, by r, and place the product under p_2; and we have seen that p'_2 is the sum of these; and thus we proceed throughout the operation, exactly according to the method of dividing by means of the detached coefficients.

In this way, $1, p_1, p_2, \ldots\ldots$, the coefficients of f_2x, are determined; and, by looking back to equation (2.), we shall see that, by operating in a similar manner on all the coefficients thus found, except R_1, we should get the coefficients of f_3x, and the second remainder R_2; and, by a continuation of the process, all the other remainders, $R_3, R_4, \ldots\ldots, R_m$ would be found; and these being the coefficients of equation $(b.)$, that equation becomes known.

It will be readily seen, that the solution of the problem in § 179. is easily derived from the present solution.

$x'^3-10x'^2-108x'+1728=0$. In this particular question, since the square of 6 is divisible by 4, the denominator of the third term, we might have taken $q=6$; and we should thus have got $x'^3-5x'^2-27x'+216=0$; an equation which would be preferable to the former as being expressed in smaller numbers. It would be easy to find the means of keeping q as small as possible; but the matter is of little importance.

183. A subject of much importance regarding equations is the *limits of their roots*. Any two numbers, one of which is greater, and the other less, than a root of an equation, are said to be *limits* of that root; the greater a *superior* limit, and the less an *inferior* one. Thus, since $\sqrt{19}$ lies between 4 and 5, if a root of an equation were $1+\sqrt{19}$, 10 and 0 would be limits of that root, as would also 7 and 4, and narrower limits still would be 6 and 5. It will thus be seen, that a root may have innumerable limits; and it is plain that the less those limits differ, the more definitely is the position of the root fixed. Many of the rules therefore that have been given on this subject are of little or no importance, giving limits that are too wide. The following are some of the most valuable in a practical point of view.

184. If two numbers x' and x'', of which x' is the greater, when substituted for x in fx, give results with contrary signs, there must lie between x' and x'' an odd number of roots of $fx=0$: but if the results have the same signs, there must lie between x' and x'' either no root, or an even number of roots. To prove this, let us put fx under the form $(x-a)(x-b)\ldots.(x-l)$, where the roots of $fx=0$ are taken in the order of their magnitudes, a being the greatest, and l the least. Now, if x' be greater than a, fx will necessarily be positive, each of its factors $x'-a$, $x'-b$, &c., being positive. Then, that the substitution of x'' for x may give a negative result for fx, an odd number of the factors, $x''-a$, $x''-b$, &c., must (§ 36.) be negative; that is, there must be only one, $x''-a$, negative; or three, as $x''-a$, $x''-b$, and $x''-c$, &c. If only $x-a$ be negative, x'' must lie between a and b; and, therefore, between it and x' there will lie one root a. If, again, three factors and no more be negative, x'' must be less than a, than b, and than c; and, therefore, between it and x' the three roots, a, b, and c, will lie: and so we might reason, if there were five factors, or any other odd number of factors, negative. To produce, however, a positive result by the substitution of x'' for x, either none of the factors must be negative, or there must be an even number of

them negative. If none of them be negative, x'' as well as x' must be greater than a; and, therefore, no root lies between x' and x''. If two factors be negative, x'' must be less than a, and less than b, but greater than c, d, &c.; and, therefore, between it and x' there will be the two roots, a and b: and similar reasoning is applicable when there are four, or any other even number of factors, negative.

Should x' be less than some of the roots, a, b, &c., suppose than each of the first m of them, we may divide

$$(x-a)(x-b)\ \ldots\ (x-l)=0$$

by the product of m factors, $x-a$, $x-b$, &c., with the sign $+$ or $-$ prefixed, as the case may require, so as to make the quotient positive: and then all the foregoing reasoning will be applicable in reference to the quotient.*

185. In the subjoined process, it is shown that the quantity marked (1.) is less than unity; m and m' being whole positive numbers, and x not less than unity: and this is a proposition which is of use regarding the limits of roots, and on some other occasions. In this, lines (2.), (3.), and (4.) are each equivalent to (1.), being derived from it by successive transformations. Line (2.) is obtained by dividing by the last term of (1.), and multiplying the result by that term: and (3.) is equivalent to (2.) according to § 137. Lastly, (4.) is derived from (3.) by dividing the numerator and denominator by $x-1$, and then by resolving the denominator $x^{m+m'}$ into the factors, x^{m-1} and $x^{m'+1}$; and, since each of the two fractional factors of the expression (4.) is less than unity, that expression must itself be less than unity; and hence the proposition is true.

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* To illustrate these principles by an example, let us take the equation $x^3-9x^2+14x+24=0$. Then, by substituting in this, first 0, and then 10 for x, we get successively for the first member 24 and 264; and these having the same sign, and the equation being of the third degree, and having therefore only three roots, there must be either no root or two roots between 0 and 10. Again, by taking $x=5$, the first member becomes -6. Now, as this differs in sign from each of the two foregoing results, there must be a root between 0 and 5, and another between 5 and 10. Then, by trying the intermediate numbers, 3 and 7, we get 12 and 24; and, as these have the same sign, there must be two roots between them. By trying, therefore, 4 and 6, we find that they are roots, as each of them makes the first member become nothing. The remaining root may be found (§ 176.) by taking 10, the sum of those already determined, from 9, the coefficient of the second term with its sign changed; that root, therefore, is -1.

$$\frac{(x-1)^m}{x^m}+\frac{(x-1)^m}{x^{m+1}}+\frac{(x-1)^m}{x^{m+2}}+\dots+\frac{(x-1)^m}{x^{m+m'}}\dots\dots(1.)$$

$$\frac{(x-1)^m}{x^{m+m'}}\cdot(x^{m'}+x^{m'-1}+x^{m'-2}+\dots+x+1)\dots\dots(2.)$$

$$\frac{(x-1)^m}{x^{m+m'}}\cdot\frac{x^{m'+1}-1}{x-1}\dots\dots\dots\dots\dots\dots(3.)$$

$$\frac{(x-1)^{m-1}}{x^{m-1}}\cdot\frac{x^{m'+1}-1}{x^{m'+1}}\dots\dots\dots\dots\dots\dots(4.)$$

186. The foregoing proposition enables us to find values of x, which will render the first term of the expression marked (1.) at the end of this §, greater than the sum of all the others, m and n being positive whole numbers, and n the greater. To effect this, let P denote the greatest in absolute value of the coefficients p_m, p_{m+1}, &c. Then, if we assume the expression (2.), the part of it after x^n is evidently greater than the corresponding part of (1.); and, therefore, if we can render x^n greater than what follows it in (2.), x^n will exceed what follows it in (1.), and in a still greater degree. Now, (2.) is transformed into (3.) by dividing by x^n, and indicating the multiplication of the result by x^n; and (4.) is obtained from (3.) by writing $(x-1)^m$ instead of P. What follows 1, in the vinculum in (4.), is the quantity which in § 185. was proved to be less than 1: and, therefore, this quantity multiplied by x^n must be less than 1 multiplied by the same, that is, than x^n itself. Hence, to render the first term in (1.) greater than all the rest, we have, as in (5.), $(x-1)^m=\mathrm{P}$; and consequently, *to find x we must take the mth root of* P, *and increase it by unity.* It is plain, however, that x may have any value we please, *greater* than the one thus assigned, as the proposition in § 185. merely requires that x shall not be less than 1.

$$x^n+p_m x^{n-m}+p_{m+1}x^{n-m-1}+\dots\dots+p_{n-1}x+p_n\dots\dots(1.)$$

$$x^n+\mathrm{P}(x^{n-m}+x^{n-m-1}+\dots\dots+x+1)\dots\dots\dots\dots(2.)$$

$$x^n\left(1+\frac{\mathrm{P}}{x^m}+\frac{\mathrm{P}}{x^{m+1}}+\dots\dots+\frac{\mathrm{P}}{x^{n-1}}+\frac{\mathrm{P}}{x^n}\right)\dots\dots\dots\dots(3.)$$

$$x^n\left(1+\frac{(x-1)^m}{x^m}+\frac{(x-1)^m}{x^{m+1}}+\dots\dots+\frac{(x-1)^m}{x^{n-1}}+\frac{(x-1)^m}{x^n}\right)(4.$$

$$(x-1)^m=\mathrm{P};\ \text{and}\ x=\mathrm{P}^{\frac{1}{m}}+1\dots\dots\dots\dots\dots\dots(5.$$

187. When an equation, $fx=0$, has one or more negative terms, and when the index of x in the first negative term differs from its highest index by m, a superior limit for the positive roots will

be found by adding 1 to the *m*th root of the negative coefficient, taken positive, which is greatest in absolute magnitude. This follows from the last §; since the number so found, or any one greater, will make the first term alone, exclusive of any positive terms that may be between it and the first negative term, exceed in amount that negative term and all the terms that may follow it, and will therefore always render fx positive, so that (§ 184.) there can be no root greater than the limit so found. Thus, in the equation, $x^3-7x-9=0$, the greatest negative coefficient is -9; and the difference between 1, the index in $7x$, the first negative term, and 3, the highest index, is 2. We take therefore the second root of 9; and, adding 1 to it, we get 4, which exceeds the greatest root of the equation.* In like manner, in the equation $x^4-3x^2-4x-3=0$, a superior limit of its roots will be $1+\sqrt{4}$, or 3; and in the equation $x^4-38x^3+210x^2+538x+289=0$, no root can exceed 39.

188. The inferior limit of the negative roots of an equation will be obtained by changing the signs of the terms occupying the even places, and finding the superior limit in the result by the last §. This, with its sign changed, will be greater in absolute magnitude than any of the negative roots; as is evident from § 178. Thus, if the equation, $x^3-x^2-14x+20=0$, be proposed, we get, by changing the alternate signs, $x^3+x^2-14x-20=0$: and (§ 187.) the superior limit of the roots of this is $1+\sqrt{20}$, or nearly $5\frac{1}{2}$ or 6. Hence, the proposed equation can have no negative root that does not lie between 0 and -6, or even between 0 and $-5\frac{1}{2}$.

189. Since (§ 176.) the last or absolute term of an equation is the product of all its roots with their signs changed, it follows, that if any of the roots be a whole number, it will be one of the factors of that term: and hence, in such cases, the roots may often be easily found by trial. Thus, in the equation,

$$x^3+4x^2+x-6=0,$$

the integral factors of the last term are 1, -1, 2, -2, 3, -3, 6, and -6: and, by taking $x=1$, fx becomes $1+4+1-6$; which, being $=0$, 1 is a root of the equation. If, again, we take $x=-1$, we get $-1+4-1-6$, or, by contraction, -4; and therefore -1 is not a root, as it does not make $fx=0$.

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* It will be found in fact, that the only real root of the equation is 3·1409233. In the next example also, the greatest root is 1·3660254; and in the third it is 30·535654.

Trying 2 in like manner, we find $fx=20$, and therefore 2 is not a root: but trying -2, we get $-8+16-2-6$, or 0; so that -2 is a root. In a similar way we should find the remaining root to be -3.

190. Since (§ 171.) if a be a root of $fx=0$, fx is divisible by $x-a$, the trials for finding the roots in the last §, and in all similar cases, are most easily effected by performing the division in the manner pointed out in § 50. Thus, the work for 1, 2, and -2, will stand as in the margin: and as the first and third give no remainders, 1 and -2 are roots; but, as in the second there is the remainder 20, 2 is not a root.

$$\begin{array}{rrr|l} 4 & 1 & -6 & 1 \\ 1 & 5 & 6 & \\ \hline 5 & 6 & 0 & \end{array}$$

$$\begin{array}{rrr|l} 4 & 1 & -6 & 2 \\ 2 & 12 & 26 & \\ \hline 6 & 13 & 20 & \end{array}$$

$$\begin{array}{rrr|l} 4 & 1 & -6 & -2 \\ -2 & -4 & 6 & \\ \hline 2 & -3 & 0 & \end{array}$$

191. When the roots of numerical equations are incommensurable, their values, though they cannot be exactly assigned in numbers, may be approximated to any degree of accuracy we please. Of all the methods that have been proposed for effecting this approximation, that which was given by the late Mr. Horner of Bath*, is much the best, combining a degree of facility and elegance belonging to no other method that has yet been given, or that is likely to be discovered. *This method, in its main feature consists in diminishing a root of the proposed equation by its first figure, according to* § 180.: *then in diminishing the corresponding root of the resulting equation, by its first figure, which is the second figure of the required root: again, in diminishing the root of the equation last obtained by its first figure, which is the third figure of the required root: and so on, till as many figures are obtained as may be considered necessary.* The process also admits of certain abbreviations, which may be regarded as a subordinate, but a very important, part of the method. The mode of proceeding and its nature will be understood from the following examples and illustrations.

Exam. 1. Required the roots of the equation,

$$x^3-4x^2-4x+20=0.$$

Here (§§ 187. and 188.) the roots must lie between 4 and -4. Now when $x=0$, $x=4$, and $x=-4$, we get 20, 4, and -92, as the respective values of fx.[*] Hence (§ 184. and note to § 178.), since there is but one permanence, there will be one root, and only one, between 0 and -4; while between 0 and 4, there may be two roots, or none. For determining whether there are two or none in the latter interval, let us take $x = 3$: then $fx = -1$: and, the sign of this differing from the results found by taking $x = 0$, and $x = 4$, there must (§ 184.) be one root between 0 and 3, and another between 3 and 4. Hence the first figure of the latter must be 3. Again, by taking $x = 2$, we get $fx = 4$; and as this differs in sign from -1, the value of fx, when $x = 3$, there must be a root between 2 and 3: the first figure, therefore of this root must be 2. To find the position of the negative root, let $x = -2$; then $fx = 4$; and as $x = -4$ gave $fx = -4$, there must be a root between -2 and -4. Taking $x = -3$, therefore, we find $fx = -3$; so that there must be a root between -2 and -3, and, accordingly, the first figure of that root must be -2.

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* Mr. Horner first published his method in the *Philosophical Transactions* for 1819.

The three following processes exhibit the computations of the roots. In the first of these, we proceed exactly in the mode pointed out in § 180., Exam. 2.: and, by taking the numbers marked with 1 subscribed, without the ciphers, we find that the equation $fx' = 0$, which has its roots less by 2 than those of the given equation, is $x'^3 + 2x'^2 - 8x' + 4 = 0$. Then, to prevent trouble in the management of decimal fractions, we annex one cipher to the coefficient of x'^2, two to that of x', and three to the remaining one. We thus get (§ 181.) the coefficients of an equation having its roots ten times as great as those of $fx' = 0$: and therefore the first figure obtained from this new equation must be taken as expressing not units but tenths. In finding this figure, which will be the first figure of the root of $x'^3 + 2x'^2 - 8x' + 4 = 0$, we are assisted by transposing 4, and dividing by $x'^2 + 2x' - 8$, as we thus get $x' = \frac{-4}{x'^2 + 2x' - 8}$, or $x' = \frac{-4}{-8}$, nearly, since, x' being small, $x'^2 + 2x'$ in the denominator, may

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[*] When $x = 0$, fx is evidently reduced to its last term. When x is 4, -4, or any other number different from zero, the value of fx is most easily found by the method of detached coefficients, as in the margin. See § 190. In a similar manner, we should find the values of fx, when $x = 3$, $x = 2$, $x = -2$, &c.

$$\begin{array}{rrr|r} -4 & -4 & 20 & 4 \\ 4 & 0 & -16 & \\ \hline 0 & -4 & 4 & \end{array}$$

$$\begin{array}{rrr|r} -4 & -4 & 20 & -4 \\ -4 & 32 & -112 & \\ \hline -8 & 28 & -92 & \end{array}$$

be rejected in roughly estimating the value of the fraction to which x' is equal. In this way we see that x' ought to be nearly equal to 0·5: on trial, however, we find that 6 is the figure which must be taken.* Then, by a process of the same kind as that in § 180., Exam. 2., we get the numbers marked with 2 subscribed, which give the equation $x''^3+38x''^2-452x''+136=0$; the roots of which are less by 6 than those of $x'^3+20x'^2-800x'+4000=0$. For the same reason as before, we add one cipher to 38, two to -452, and three to 136: and then, by dividing 136000 by -45200 with its sign changed, we get 3 as the next figure, with which we proceed as we did regarding the last figure; and we get, as the coefficients of the terms after the first in the next equation, 389, -42893, and 3847. To the first of these we might annex one cipher, to the second two, and to the third three; and, by proceeding as before, we might find the next figure, and the coefficients of the equation which would give the figure after it: and, by continuing to work in a similar way, we might evolve figure after figure, till we should attain any degree of accuracy that might be required. It is plain, however, that, in this way, the coefficients would rapidly increase in magnitude, and that the operation would thus become very laborious: and therefore we ought to consider, whether we cannot fall on some expedient for abridging the labour. We are enabled to effect this object in the most advantageous manner possible, on the same principle that is employed in contracting the division of decimal fractions in common arithmetic. In employing this principle in the present example, instead of annexing ciphers, we point off one figure in the last column but one, and two in the column preceding it. Then, as 4289 is not contained in 3847, a cipher is annexed to the root; and one figure more is pointed off in the last column but one, and two in the preceding one, which is thus exhausted, even with a cipher prefixed. After this, the work proceeds exactly according to the contracted method of the division of decimal fractions.† (See *Arithmetic*, p. 111.)

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* This is illustrated in the subjoined operations. When we take 5, 6, and 7, we find the corresponding values of the function to be 625, 136, and -277; and therefore (§ 184.) the required value must lie between 6 and 7, on account of the change in the signs of the results.

$$\begin{array}{rrr|l|rrr|l|rrr|l} 20 & -800 & 4000 & 5 & 20 & -800 & 4000 & 6 & 20 & -800 & 4000 & 7 \\ \underline{5} & \underline{125} & \underline{-3375} & & \underline{6} & \underline{156} & \underline{-3864} & & \underline{7} & \underline{189} & \underline{-4277} & \\ 25 & -675 & 625 & & 26 & -644 & 136 & & 27 & -611 & -277 & \end{array}$$

† To see clearly the nature of the contracted process, the student

-4	-4	20 (2·630897
2	-4	-16
-2	-8	$_{1}4000$
2	0	-3864
0	$-_{1}800$	$_{2}136000$
2	156	-132153
$_{1}20$	-644	$_{3}3847$
6	192	-3431
26	$-_{2}45200$	416
6	1149	-386
32	-44051	-30
6	1158	-30
$_{2}380$	$-_{3}42893$	
3	· · · ·	
383		
3		
386		
3		
$0_{3}389$		
· ·		

In the following process for finding the second root, after the numbers marked with 4 subscribed have been found, the contraction is commenced by cutting off one figure in the last column but one, and two in the column before it. When figures are thus cut off, we merely consider what we ought to *carry* from them, if they were retained, but in every other respect we neglect them. After we have cut off figures twice in this way in the present example, the first column is exhausted; and, therefore, the rest of the work is completed according to the contracted process for the division of decimals.

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should work this example, and perhaps one or two others, at full length, so as to find four or five places of decimals; and then, by drawing a vertical line in each column (see *Arithmetic*, p. 111.), he will separate the part retained from that which is rejected in the contracted process. It may be remarked, that, in addition to the valuable method of contraction that has been pointed out, various other abbreviations will, on many occasions, readily suggest themselves to the student. Few of these, however, will be found to be of importance; as, in most instances, they do little more than shorten the process to the eye, without really abridging the labour.

−4	−4	20 | 3·525428
3	−3	−21
−1	−7	$-_{2}1000$
3	6	875
2	$-_{2}100$	$-_{3}125000$
3	275	97608
$_{2}50$	175	$-_{4}27392$
5	300	25220
55	$_{3}47500$	−2172
5	1304	2031
60	48804	−141
5	1308	101
$_{3}650$	$_{4}50112$	−40
2	33·	40
652	5044	
2	33	
654	5077	
2	...	
$0_{4}656$		

The operation in the next page for finding the negative root is in every respect similar to the others.*

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* The sum of the three roots found in the text is exactly 4; and as this is the same as the coefficient of the second term of the given equation with its sign changed, which (§ 176.) it ought to be, we have strong reason to presume that the roots are correct, as far as they are carried, unless perhaps in their last figures, in which there might be small errors balancing each other. For the benefit of the student, the three roots have been computed independently of one another. When any one of them, however, was found, it might have been attached to x with its sign changed; then by dividing fx by the expression thus found, and putting the quotient equal to 0, a quadratic equation would be obtained, the roots of which (§ 174.) would be the remaining roots of $fx=0$. The division in reference to the root first found, is annexed by the method of detached coefficients.

−4	−4	20	| 2·630897
2·630897	−3·601969	−19·999997	
−1·369103	−7·601969	0·000003	

Were the root quite accurate, there would be no remainder; and though there is the remainder, 0·000003, the minuteness of this shows that

-4	-4	$20 \mid -2{\cdot}156325$
-2	12	-16
-6	8	${}_{2}4000$
-2	16	-2501
-8	${}_{2}2400$	${}_{3}1499000$
-2	101	-1327375
$-{}_{2}100$	2501	${}_{4}171625$
-1	102	-162780
-101	${}_{3}260300$	8845
-1	5175	-8158
-102	265475	687
-1	5200	-544
$-{}_{3}1030$	${}_{4}270675$	143
-5	$63\cdot$	-136
-1035	27130	7
-5	63	
-1040	27193	
-5	$\dots$	
$-{}_{4}1045$		
. .		

Exam. 2. Resolve the equation $x^4+x^2-8x-15=0$.

The coefficients of this incomplete equation are 1, 0, 1, -8, and -15. Now, if we take $-$ before 0, we shall have three changes of signs; and, by Descartes's rule, we might have three positive roots. This cannot be so; however, since if we take $+$, as we are equally entitled to do, we have only one change of signs, and therefore there cannot be more than one positive root. In like

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2·630897 is extremely nearly true. We have, therefore, the quadratic equation, $x^2-1{\cdot}369103x-7{\cdot}601969=0$; by the resolution of which we should get exactly the same values as those found above for the second and third roots.

According to § 178., if a positive root of the equation, $x^3+4x^2-4x-20=0$, be found, and if its sign be changed, the result will be a negative root of the proposed equation. Hence, by taking, as headings, 4, -4, and -20, we should find 2·156325 as a root; the negative of which is the only negative root of the proposed equation. Some will, perhaps, prefer this method of finding negative roots to the direct one employed above. In general, however, no advantage is gained in using one of the methods in preference to the other.

manner, by taking + before 0, we have three permanences, but by taking — we have only one; and therefore there cannot be more than one negative root. The equation therefore cannot have more than two real roots, one positive and the other negative; and these it is found to have, and (§§ 187. and 188.) they lie between 4 and —3. Then, by taking *x* successively equal to 4, 0, and —3, we get 225, —15, and 99; and by farther trials we find that the values of *x* lie between 2 and 3, and between —1 and —2. The computations of the two roots are exhibited in the subjoined processes.

0	1	−8	−15	2·302775637
2	4	10	4	
2	5	2	−$_{2}$110000	
2	8	26	108741	
4	13	$_{2}$28000	−$_{3}$12590000	
2	12	8247	9066704	
6	$_{2}$2500	36247	−3523296	
2	249	9021	3179547	
$_{2}$80	2749	$_{3}$45268000	−343749	
3	258	6552·	318129	
83	3007	4533352	−25620	
3	267	6556	22724	
86	$_{3}$327400	4539908	−2896	
3	2·	230·	2727	
89	3276	454221	−169	
3	2	230	136	
$_{3}$920	3278	454451	−33	
·	2	2·	32	
	3280	45447		
	· ·	2		
		45449		
		· · · ·		

In the foregoing operation, the full number of ciphers is added twice over, and we are thus enabled to get nine places of decimals, which may all be depended on except the last, or at most the last two; a degree of accuracy which is seldom necessary.

In the process in the next page, had the contraction been commenced with the coefficients marked with 3 subscribed, since the last but one of them, —19388, admits of having four figures cut off without being annihilated, we should have got, besides the decimal figure 3 already found, four others, the last of which, however, could not be depended on. Should this number of places be considered too small, and should we wish to get not four others, but two, we may proceed as in the operation, annexing two ciphers in the last column, one in the preceding column, and none in the one before that; and, lastly, pointing off one figure in the first column. Then, as the number in the third column is not contained in the one in the fourth, we put a cipher in the root; and, after that, cutting off ciphers, we proceed as in the former operations. The root so found is true in all its figures except the last, which, as in the other root, ought to be 6.

0	1	−8	−15	−1·3027757
−1	1	−2	10	
−1	2	−10	$-{}_{2}50000$	
−1	2	−4	49461	
−2	4	$-{}_{2}14000$	$-{}_{3}53900$	
−1	3	−2487	38820	
−3	${}_{2}700$	−16487	−15080	
−1	129	−2901	13608	
$-{}_{2}40$	829	$-{}_{3}193880$	−1472	
−3	138	−22·	1361	
−43	967	−19410	−111	
−3	147	−22	97	
−46	${}_{3}1114$	−19432	−14	
−3	· ·	−1·	13	
−49		−1944		
−3		−1		
$-0_{3}52$		−1945		
· ·		· · ·		

By attaching the roots that have been found to x with their signs changed, and taking the product of the two expressions, we get x^2-x-3; and if the first member of the given equation be divided by this, and the quotient be put equal to 0, there is obtained $x^2+x+5=0$; an equation, the roots of which (§ 174.) are the remaining roots of the given equation. Resolving this therefore, we find the two imaginary roots to be $\frac{1}{2}(-1+\sqrt{-19})$, and $\frac{1}{2}(-1-\sqrt{-19})$. It thus appears, that the first member of the proposed equation is the product of the quadratic factors, x^2-x-3 and x^2+x+5.

Exam. 3. **Required the fifth root of 123456789.**

The effecting of what is here required is nothing else than the resolution of the equation $x^5=123456789$, or $x^5-123456789=0$. Now, in binomial equations, such as the present, we can find the positions of the roots more simply than in other cases. Thus, (§ 98.) 40^5 being $=4^5\times 10^5$, or $4^5\times 100000$, and $50^5=5^5\times 100000$, it is plain, that the fifth power of any number between 40 and 50, being greater than one of these and less than the other, will lie between 4^5 and 5^5 with five ciphers annexed to each. Hence therefore, in the operation, five figures are cut off towards the right hand; and then 1234, the number remaining to the left, being between 1024, the fifth power of 4; and 3125, the fifth power of 5, the root must lie between 40 and 50.* We take 4 therefore, as its first figure; and placing as headings to the first four columns, four zeros, the coefficients of x^4, x^3, x^2, and x, we proceed exactly as in the other examples.

0	0	0	0	−1234′56789	41·524366
4	16	64	256	1024	
4	16	64	256	$-{}_{2}21056789$	
4	32	192	1024	13456201	
8	48	256	${}_{2}12800000$	$-{}_{3}7600588$	
4	48	384	656201	7238800	
12	96	${}_{2}640000$	13456201	−361788	
4	64	16201	672604	296898	
16	${}_{2}16000$	656201	${}_{3}14128805$	−64890	
4	201	16403	34880 ·	59448	
${}_{3}200$	16201	672604	1447760	−5442	
1	202	16606	35300	4460	
201	16403	${}_{3}689210$	1483060	−982	
1	203	84 ·	143 ·	892	
202	16606	6976	148449	−90	
1	204	84 ·	143	89	
203	$0_{3}16810$	7060	148592		
1	· ·	84	3 ·		
204		7144	14862		
1		· ·	3		
$0_{3}205$			14865		
·			· · ·		

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* This might be easily generalised, and it would appear, that in extracting the *n*th root of a number, if we cut off, as often as possible, *periods*, as they have been called, of *n* figures each, the first figure of the root will be that whose *n*th power is either equal to the left-hand period, complete

Exercises. Required the roots of the following equations.

1. $x^3+7x-3=0$. *Ans.* $x=\cdot418128$.
2. $x^3-2x^2+3x-4=0$. *Ans.* $x=1\cdot650629$.
3. $x^3+2x^2+3x+4=0$. *Ans.* $x=-1\cdot650629$.
4. $x^4-2x^3+3x-20=0$.
Ans. $2\cdot648688$, and $-1\cdot876837$. The rest are imaginary.
5. $x^4-2x^3-3x^2-4x+5=0$.
Ans. $x=3\cdot182478$, and $x=0\cdot728727$.
6. $x^5-7x^4+15x^3-58x^2+44x-300=0$. *Ans.* $x=6\cdot119538$.
7. $x^2+x-1=0$. *Ans.* $x=0\cdot618034$, and $x=-1\cdot618034$.
8. $x^3+x^2+x-1=0$. *Ans.* $x=0\cdot543689$.
9. $x^4+x^3+x^2+x-1=0$.
Ans. $x=0\cdot518790$, and $x=-1\cdot290649$.
10. $x^5+x^4+x^3+x^2+x-1=0$. *Ans.* $0\cdot508660$.
11. $x+\sqrt[3]{(x-5)}-10=0$.* *Ans.* $8\cdot4840198$.

192. If an equation, $fx=0$, have m roots, each equal to a, fx (§ 172.) will have $(x-a)^m$ as a factor. Conversely, if fx contain the factor $(x-a)^m$, $fx=0$ has m roots each equal to a; for (§ 171.) taking a as a root, if we were to divide by $x-a$, the result would contain the factor $(x-a)^{m-1}$; and that result being put equal to nothing, a would also be a root of the equation so obtained; and therefore we might again divide by $x-a$, and so on. It thus appears that if $fx=0$ have m roots each equal to a, and if fx be divided by $x-a$, the quotient and fx will have the common factor $(x-a)^{m-1}$: and consequently, if we can, directly or indirectly, perform the division, and then find the greatest common divisor of the quotient and fx, that divisor will be $(x-a)^{m-1}$. As the root is not known, however, we cannot perform the division *directly*: let us therefore endeavour to find the quotient in some other way. To enable us to effect this, let us write

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or incomplete, or is less than it, but which approaches it most nearly. Thus, if the cube root of 31460259 were required, we should divide it into periods, as follows; 31′460′959: and the first figure of the root would be 3, the root of 27, the greatest cube contained in the first period, 31. The work would then proceed in the usual manner: and thus the extraction may be effected in all similar cases.

* This may be solved by freeing it (§ 110.) of radicals, and then proceeding in the usual way; or, perhaps preferably, by putting $\sqrt[3]{(x-5)}=y$, so that the given equation may become $y^3+y-5=0$. Then, the value of y being obtained from this latter equation, that of x will be found by cubing it, and adding 5 to the result.

$$fx=x^n+p_1x^{n-1}+p_2x^{n-2}+\dots\dots+p_n=0,$$

under the form,

$$(x^n-a^n)+p_1(x^{n-1}-a^{n-1})+p_2(x^{n-2}-a^{n-2})+\dots+p_{-1}(x-a)$$
$$+a^n+p_1a^{n-1}+p_2a^{n-2}+\dots+p_{n-1}a+p_n.$$

Now (§ 58.) the quantity in each vinculum is divisible by $x-a$: and, since a is a root of $fx=0$, the second line (§ 171.) must be equal to nothing, since it is what fx becomes, when x is taken equal to a. If we divide, therefore, the first line by $x-a$, the result will be the quotient that would be obtained by dividing fx by the same. Hence, if we divide the quantity in each vinculum by $x-a$, according to § 58., and multiply the second quotient by p_1, the third by p_2, the fourth by p_3, &c., we shall find the entire quotient to be

$$\begin{array}{r|r|r} x^{n-1}+a & x^{n-2}+a^2 & x^{n-3}+\dots+a^{n-1} \\ +p_1 & +p_1a & \dots+p_1a^{n-2} \\ & +p_2 & \dots+p_2a^{n-3} \\ & & \dots\dots\dots \\ & & +p_{n-1} \end{array}$$

193. The quotient obtained in the last § is true universally, whatever may be the nature of the roots; as in finding it no supposition has been made regarding a, except that it is a root of the equation. If, however, $fx=0$ have two or more roots, each equal to a, that quotient will become nothing, if x be changed into a. Let that change be made, and the first line of the quotient will become $a^{n-1}+a^{n-1}+\dots\dots+a^{n-1}$, or na^{n-1}, the number of terms being plainly n. In like manner, the second line would become $(n-1)p_1a^{n-2}$, the third $(n-2)p_2a^{n-3}$; and so on: and the entire quotient becomes simply

$$na^{n-1}+(n-1)a^{n-2}+(n-2)a^{n-3}+\dots+p_{n-1}=0.$$

Now, this is the same that would be obtained by changing x into a in $nx^{n-1}+(n-1)x^{n-2}+\dots\dots+p_{n-1}=0$: and, as the change of x into a satisfies this equation by making the first member equal to nothing, a must be a root of it; and therefore its first member, which may be denoted by $f'x$, is divisible, in common with fx, by $x-a$.

194. By comparing $f'x$ with x, we shall find that the latter may be derived from the former by the following extremely simple and easy rule: Multiply each term of fx by the index in the same term, and diminish that index by unity.*

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* It will be seen, that in this process the last term p_n disappears. Even

195. We have it now in our power to discover whether any equation $fx=0$ has equal roots, and if so, to determine them, as we have merely to find $f'x$ by the rule given in the last § ; and then, by § 81. or § 82. to find the greatest common divisor of fx and $f'x$: if this be unity, there are no equal roots ; but if it be $(x-a)^m$, there are $m+1$ of them, each equal to a. It is plain, also, from exactly similar considerations, that in general, if the common divisor should be found to be $(x-a)^m(x-b)^k \ldots$, there would be $m+1$ roots each equal to a, $k+1$ each equal to b, and so on.

Exam. 1. Find whether the equation $x^3-7x^2+16x-12=0$ has equal roots ; and if so, determine them. Here, by § 194., we find $f'x$ to be $3x^2-14x+16$; and, by § 81. or § 82., the greatest common measure of this and of $x^3-7x^2+16x-12$ is found to be $x-2$. Hence the equation has two roots, each equal to 2. Then, by dividing the first member of the given equation by $(x-2)^2=x^2-4x+4$, we get $x-3$; and therefore the remaining root is 3.

Exam. 2. Find whether the equation,

$$x^4-4x^3-2x^2+12x+9=0,$$

has equal roots ; and, if so, determine them. Here, we have $f'x=4x^3-12x^2-4x+12$; and, by dividing this by 4 for the sake of a simplification, and by finding the greatest common divisor of the quotient and fx, we get x^2-2x-3 : which (§ 161.) is the same as $(x-3)(x+1)$. Hence the proposed equation has two roots each equal to 3, and two others each equal to -1 ; and being of the fourth order, it has no others.

Exercises. Find the roots of the following equations, which have equal roots.

1. $x^3+x^2-x-1=0$. *Ans.* -1, -1, 1.
2. $x^3-9x^2+27x-27=0$. *Ans.* 3, 3, 3.
3. $x^4-2x^3-3x^2+4x+4=0$. *Ans.* 2, 2, -1, -1.
4. $x^5-15x^3-10x^2+60x+72=0$. *Ans.* 3, 3, -2, -2, -2.
5. $x^5-6x^4+3x^3+46x^2-108x+72=0$. *Ans.* 2, 2, 2, 3, -3.

with regard to it, however, the rule will be seen to hold, if it be put under the form $p_n x^0$, as the multiplication by the index 0 will destroy the term. The function $f'x$ is called by the French mathematicians the *derived function* (*fonction dérivée*, or simply *dérivée*) ; and it is the same as what is generally called the *differential coefficient* by the writers on the differential calculus. Sometimes also it is called the *limiting equation*.

196. From what has been established regarding equations having equal roots, it appears that their roots may all be obtained by the resolution of equations of lower degrees than their own; and the same is the case regarding what are called *reciprocal equations.* We may now, therefore, enter on a short consideration of equations of this kind. A *reciprocal equation* is one of such a kind that if a be one of its roots, the reciprocal of a will be another. For discovering the nature and form of such equations, it is necessary to consider one of an odd and one of an even degree. Let us therefore consider, first,

$$fx = x^5 + p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5 = 0,$$

an equation of the fifth degree. Now, by the definition given above, this must hold equally, if x be changed into its reciprocal x^{-1}, so that $x^{-5} + p_1x^{-4} + p_2x^{-3} + p_3x^{-2} + p_4x^{-1} + p_5 = 0$. Multiply the original equation by p_5; multiply also the last result by x^5, reversing the order of the terms: then

$$p_5x^5 + p_5p_1x^4 + p_5p_2x^3 + p_5p_3x^2 + p_5p_4x + p_5^2 = 0, \text{ and}$$
$$p_5x^5 + p_4x^4 + p_3x^3 + p_2x^2 + p_1x + 1 = 0.$$

Now, these will be identical, if the coefficients of the like powers of x be equal. Hence we have $p_5^2 = 1$, and therefore $p_5 = \pm 1$. Taking $p_5 = 1$, we get $p_4 = p_1$, and $p_3 = p_2$: but if we take $p_5 = -1$, we find $p_4 = -p_1$ and $p_3 = -p_2$. Hence fx takes either of the forms,

$$x^5 + p_1x^4 + p_2x^3 + p_2x^2 + p_1x + 1 = 0 \ldots (1.)$$
$$\text{and } x^5 + p_1x^4 + p_2x^3 - p_2x^2 - p_1x - 1 = 0 \ldots (2.)$$

It thus appears that, throughout, the coefficients of terms equally remote from the extreme ones, are either equal and have like signs, or are equal but have contrary signs; and this is always the case when the equation is of an odd degree, as would be shown by operating, as above, on the general equation,

$$x^{2m+1} + p_1x^{2m} + \ldots\ldots + p_{2m+1} = 0.$$

Again, if we take

$$fx = x^6 + p_1x^5 + p_2x^4 + p_3x^3 + p_4x^2 + p_5x + p_6 = 0,$$

and proceed in a similar manner, it will appear that fx may be of either of the two forms,

$$x^6 + p_1x^5 + p_2x^4 + p_3x^3 + p_2x^2 + p_1x + 1 = 0 \ldots\ldots (3.) \text{ and}$$
$$x^6 + p_1x^5 + p_2x^4 - p_2x^2 - p_1x - 1 = 0 \ldots\ldots (4.)$$

Hence, the coefficients of any two terms equally distant from the extreme ones are equal; and they have either both the same sign, as in (3.); or opposite signs as in (4.); also, when the signs are opposite, as in (4.), the middle term is wanting. This would be shown to hold regarding all reciprocal equations of an even degree, by employing the general equation,

$$x^{2m}+p_1x^{2m-1}+\ \ldots\ldots,\ +p_{2m}=0.$$

197. Every reciprocal equation of an odd degree has one of its roots equal to 1, when its first and last terms have contrary signs; and equal to -1 when they have the same sign. Thus, equation (2.) by connecting the first and last terms, the second and last but one, &c., may be put under the form,

$$(x^5-1)+p_1x(x^3-1)+p_2x^2(x-1)=0;$$

which (§ 58.) is divisible by $x-1$; and therefore 1 is a root: and in a similar manner it would be shown, that equation (1.) is divisible by $x+1$, and therefore -1 is a root. Hence, by dividing a reciprocal equation of an odd degree by $x+1$ or or $x-1$, as the case may require, and by putting the quotient equal to 0, we shall have an equation, the roots of which will be the remaining roots of the reciprocal one. The equation thus obtained will also be a reciprocal one; as is plain, from considering the quotients found by dividing $x^{2n+1}+1$ by $x+1$, and $x^{2n+1}-1$ by $x-1$; in which it appears from § 60., that each of the terms will have 1 or -1 as coefficient; and that, when the first and last have the same sign, so will also those that are equally distant from them; but that if the first and last have contrary signs, the signs of those equally distant from them will also be unlike. It only remains therefore, that we find the method of resolving reciprocal equations of an even order; and the following examples will show how that is effected.

Exam. 1. Let it be required to resolve the reciprocal equation, $x^4-3x^3-2x^2-3x+1=0$, which is of the same form as equation (3.).

$$(x^4+1)-3(x^3+x)-2x^2=0\ \ldots\ldots\ (1.)$$
$$(x^2+x^{-2})-3(x+x^{-1})-2=0\ \ldots\ (2.)$$
$$x+x^{-1}=y\ \ldots\ldots\ldots\ (3.)$$
$$x^2+2+x^{-2}=y^2\ \ldots\ldots\ldots\ (4.)$$
$$x^2+x^{-2}=y^2-2\ \ldots\ldots\ldots\ (5.)$$
$$y^2-2-3y-2=0\ \ldots\ldots\ldots\ (6.)$$
$$y^2-3y=4\ \ldots\ldots\ldots\ (7.)$$
$$y=4,\text{ and }y=-1\ \ldots\ldots\ldots\ (8.)$$
$$x=\tfrac{1}{2}\{y\pm\sqrt{(y^2-4)}\}\ \ldots\ldots\ldots\ (9.)$$
$$x=2\pm\sqrt{3}\ \ldots\ldots\ldots\ (10.)$$
$$x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}\ \ldots\ldots\ldots\ (11.)$$

In the annexed work, equation (1.) is obtained by connecting the first and last terms of the given equation, and also those next them; and (2.) is derived from this by dividing by x^2. In (3.), $x+x^{-1}$ is assumed equal to y; and from it (4.) and (5.) are derived by squaring and transposing. Equation (6.) is obtained by substituting in (2.) y^2-2 and y for their equals in (5.) and (3.); (7.) is derived from this by transposition; and (8.) is found by resolving the quadratic equation in (7.). We get (9.) from (3.) by multiplying by x, by transposition, and by resolving the quadratic so obtained. In the last place, (10.) and (11.) are obtained from (9.) by substituting for y the values found in (8.).

Exam. 2. Resolve the equation, $x^4-3x^3+3x-1=0$, which is of the same form as (4.).

Here, (1.) is obtained by connecting the first and last terms, and also the others: and (2.) is the same as this, resolved into factors. In (3.) and (4.) these factors are put separately equal to nothing, as each of these assumptions satisfies (2.). Then (5.) and (6.) are found by resolving (3.) and (4.).

$$(x^4-1)-3x(x^2-1)=0 \ldots (1.)$$
$$(x^2+1-3x)(x^2-1)=0 \ldots (2.)$$
$$x^2-1=0 \ldots\ldots (3.)$$
$$x^2+1-3x=0 \ldots\ldots (4.)$$
$$x=\pm 1 \ldots\ldots (5.)$$
$$x=\tfrac{1}{2}(3\pm\sqrt{5}) \ldots\ldots (6.)$$

Exam. 3. Resolve the equation $8x^6-16x^4-25x^3-16x^2+8=0$, which is of the third form, the coefficients of x^5 and x being each nothing.

Here, equation (1.) is obtained by connecting the terms, as in the former examples, and (2.) is found from it by dividing by x^3. Equation (3.) is got by multiplying the members of equation (5.) in Ex. 1. by $x+x^{-1}$ and y respectively: and from (3.) we get (4.) by using y instead of $x+x^{-1}$, and transposing it. By substituting the value just found in (2.), and also y in the same

$$8(x^6+1)-16(x^4+x^2)-25x^3=0 \ldots (1.)$$
$$8(x^3+x^{-3})-16(x+x^{-1})-25=0 \ldots (2.)$$
$$x^3+x^{-3}+x+x^{-1}=y^3-2y \ldots\ldots (3.)$$
$$x^3+x^{-3}=y^3-3y \ldots\ldots (4.)$$
$$8(y^3-3y)-16y-25=0 \ldots\ldots (5.)$$
$$8y^3-40y-25=0 \ldots\ldots (6.)$$
$$y=\tfrac{1}{2}y' \ldots\ldots (7.)$$
$$y'^3-20y'-25=0 \ldots\ldots (8.)$$
$$y'=5, \text{ and } y'=\tfrac{1}{2}(-5\pm\sqrt{5}) \ldots\ldots (9.)$$
$$y=\tfrac{5}{2}, \text{ and } y=\tfrac{1}{4}(-5\pm\sqrt{5}) \ldots\ldots (10.)$$
$$x=2, \text{ and } x=\tfrac{1}{2} \ldots\ldots (11.)$$
$$x=\tfrac{1}{8}\{-5\pm\sqrt{5}\pm\sqrt{(-34\mp 10\sqrt{5})}\} (12.)$$

for $x+x^{-1}$, we get (5.), the modification of which gives (6.). To transform (6.) into an equation having the coefficient of its first term 1, we change (§ 182.) y into $\frac{1}{2}y'$, as in (7.), and thus (6.) becomes (8.). Now, the divisors of the last term of this being 1, -1, 5, -5, 25, and -25, we readily find (§ 189.) that one value of y' is 5. Then, dividing the first member of (8.) by $y'-5$, putting the quotient equal to nothing, and resolving the equation so obtained, we get the values of y' in (9.); and, according to the assumption in (7.) we find the values of y as in (10.). Equation (11.) contains the values of x found by taking $y=\frac{5}{2}$ in equation (9.), Exam. 1.; and those in (12.) by taking y in the same equal to $\frac{1}{4}(-5\pm\sqrt{5})$.

198. From pursuing the mode of investigation employed in Exam. 2., it will appear that every equation of the form of equation (4.) in § 196. has one root equal to 1, and another to -1; and that, by division by x^2-1, an equation two degrees lower will be obtained, the roots of which will be the remaining roots of the original equation; and as this will be a reciprocal equation, its roots will be found by means of an equation of half its dimensions. Hence, an equation of the nth degree of the form (4.), requires only the resolution of an equation of the order $\frac{1}{2}(n-2)$. It is plain, also, that an equation of the form (1.) or (2.) in § 196. requires only the resolution of an equation of the order $\frac{1}{2}(n-1)$; while one of the form (3.) is resolved by means of an equation of the order $\frac{1}{2}n$.

199. *Binomial equations*, that is, equations of the form $x^n\pm p_n=0$, are a remarkable class of reciprocal equations, which may now be briefly considered. If we put the nth root of p_n equal to p', so that $p_n=p'^n$, we shall have $x^n\pm p'^n=0$. Then, changing x into $p'z$, we get $p'^nz^n\pm p'^n=0$; whence $z^n\pm1=0$. Hence, therefore, if we can find the values of z in this equation, we shall obtain those of x, since $x=p'z$.

200. Let us first consider the case in which $z^n-1=0$, n being an odd number. Hence, $z^n=1$, and therefore, one value of z is 1, since the nth power of 1 is 1. It is plain also, that there can be no other real root, since the nth power of no other real number whatever, such as 2, -2, $\frac{1}{2}$, &c., can be 1: and consequently, the equation must have $n-1$ imaginary roots. To find these, divide (§ 58.) z^n-1 by $z-1$: then, by putting the quotient equal to nothing, we get

$$z^{n-1}+z^{n-2}+z^{n-3}+\ \dots\ +z+1=0;$$

a reciprocal equation of the third form, the resolution of which will give the values of z.

201. If the equation be $z^n+1=0$, n being still odd, we shall have merely to change the sign of z in the last §, as we shall thus get $-z^n-1=0$, or $z^n+1=0$. Hence, if we find, as in the last §, the roots of $z^n-1=0$, these (§ 178.) with their signs changed will be the roots of $z^n+1=0$.

Exam. 4. Required the three cube roots of 1. This problem is the same as to resolve the equation $z^n=1$, or $z^n-1=0$; and, n being 3, and one value of z being 1, the depressed equation in § 173. becomes $z^2+z+1=0$; the roots of which are $\frac{1}{2}(-1+\sqrt{-3})$ and $\frac{1}{2}(-1-\sqrt{-3})$. These, therefore, and 1 are the three required roots. It follows, also, from the last §, that the three cube roots of -1 are -1, $\frac{1}{2}(1-\sqrt{-3})$, and $\frac{1}{2}(1+\sqrt{-3})$.

The cube roots of any other number $\pm a$, or $a\times\pm 1$, will be found (§ 98.) by multiplying the *arithmetical* cube root of a, into the three cube roots of 1 or -1, as the case may be. Thus, the cube roots of 125 are 5, $\frac{5}{2}(-1+\sqrt{-3})$, and $\frac{5}{2}(-1-\sqrt{-3})$; and the cube roots of -27 are -3, $\frac{3}{2}(1-\sqrt{-3})$, and $\frac{3}{2}(1+\sqrt{-3})$.

202. If the equation be $z^n-1=0$, and n even, we shall have the square root of z^n equal to 1 or -1. Thus, if $z^6-1=0$, we get $z^3=1$, and $z^3=-1$; and therefore the six roots of $z^6=1$ are the same as the roots of $z^3=1$ and $z^3=-1$, taken together.

203. If the equation be $z^n+1=0$, with n even, the roots will be obtained by resolving it directly as a reciprocal equation; and it is plain that all the roots will be imaginary, as -1 has no real even root. If, for instance, we have $z^4+1=0$, we get $z^2+z^{-2}=0$; which, by taking $z+z^{-1}=v$, becomes $v^2-2=0$; whence $v=\pm\sqrt{2}$. Hence, $z+z^{-1}=\pm\sqrt{2}$; whence, by resolving the equation, we get $z=\frac{1}{2}(\pm\sqrt{2}\pm\sqrt{-2})$; an expression which contains the four fourth roots of -1.*

Exercises. Find the roots of the following equations.

1. $3x^4+2x^3-34x^2+2x+3=0$.

Ans. $x=3^{\pm 1}$, and $x=-2\pm\sqrt{3}$.

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* Those who wish for further information regarding binomial equations, may have recourse to Gauss's *Disquisitiones Arithmeticæ*; Legendre, *Théorie des Nombres*; and Lagrange, *Résolution des Equations Numériques*. The trigonometrical resolution of such equations will be found in the Author's *Treatise on the Differential and Integral Calculus*, Section XV.

2. $2x^5-x^4-4x^3-4x^2-x+2=0$.

Ans. $x=-1$, $x=2^{\pm 1}$, and $x=\frac{1}{2}(-1\pm\sqrt{-3})$.

3. $16x^6-64x^5-x^4+x^2+64x-16=0$.

Ans. $x=\pm 1$, $x=4^{\pm 1}$, and $x=\frac{1}{8}(-1\pm\sqrt{-63})$.

4. $8x^8+16x^6-125x^5+125x^3-16x^2-8=0$.

Ans. $x=\pm 1$, and $x=\frac{1}{4}\{5\sqrt[3]{1}\pm\sqrt{[25(\sqrt[3]{1})^2-16]}\}$.*

5. Find all the fifth roots of 1024.

Ans. 4, and $-1\pm\sqrt{5}\pm\sqrt{(\pm 2\sqrt{5}-10)}$.

6. Resolve the equation $x^4+1=0$. *Ans.* $(\frac{1}{2}\pm\sqrt{2}\pm\sqrt{-2})$.

CHAPTER XI.

INDETERMINATE COEFFICIENTS AND THE BINOMIAL THEOREM.

204. The method of indeterminate coefficients is of much use in various investigations. The general principle of this method consists in the assuming of unknown coefficients for the required quantity: then, by finding, according to the nature of the investigation, two series or expressions of the same form, which are to be identical, we render them so by taking their corresponding terms, or the coefficients of those terms, equal; and we thus obtain a series of equations, which give the values of the assumed coefficients. As a simple example, let it be required to divide a by $1+2x+x^2$, according to this method, taking, for simplicity, 1 as the leading term of the divisor. By carrying out the division in the ordinary way, so as to get two or three terms of the quotient, and considering the relations of those terms, we should see that the first term would not contain x, while the other terms would contain its successive powers, with coefficients independent of that quantity. Let us assume, therefore,

$$\frac{a}{1+2x+x^2}=A_0+A_1x+A_2x^2+A_3x^3+\ldots+A_nx^n+\&c.\ldots(1.);$$

where A_0, A_1, …., A_n are the coefficients which are to be determined. Now, if the two members be multiplied by the same quantity the products must be equal. Let them be multiplied by $1+2x+x^2$, so as to obtain quantities free of fractions, and we shall get

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* The six roots comprehended in this expression will be found by employing all the cube roots of 1 found in Exam. 4. p. 199. Two of the roots so found will be $2^{\pm 1}$. The rest are imaginary.

$$\left.\begin{array}{r|l|l|l|l}
a=A_0+\ A_1 & x+\ A_2 & x^2+\ A_3 & x^3+\ldots+\ A_n & x^n+\&c.\\
+2A_0 & +2A_1 & +2A_2 & +\ldots+2A_{n-1} & +\&c.\\
 & +\ A_0 & +\ A_1 & +\ldots+\ A_{n-2} & +\&c.
\end{array}\right\}.(2.)^*$$

Now, whatever may be the value of x, the members of this equation will be rendered the same, if $A_0=a$, and if the coefficients of the several powers of x be taken equal to 0.† Since therefore $A_0=a$, by taking the next coefficient we get $A_1+2A_0=0$, or $A_1+2a=0$; whence $A_1=-2a$. In like manner we have $A_2+2A_1+A_0=0$, or $A_2-4a+a=0$; and therefore $A_2=3a$: and from the next coefficient we should find $A_3=-4a$, and thus we might proceed, as far as we please. From the general term we get $A_n=-2A_{n-1}-A_{n-2}$: and hence it appears, that each coefficient after the first is found from the two immediately preceding it, by adding the first of them to twice the second, and changing the sign of the sum. By introducing the values thus found in (1.), we get for the required quotient, $a-2ax+3ax^2-4ax^3+\&c.$; the same that would be found by division.

As another example, let it be required to find the square root of a^2+x by the method of indeterminate coefficients.

Here we assume $\sqrt{(a^2+x)}=A_0+A_1x+A_2x^2+A_3x^3+\&c.$: and by squaring both members, so as to get quantities of the same form, we obtain

$$\begin{array}{r|l|l}
a^2+x=A_0{}^2+2A_0A_1x+2A_0A_2 & x^2+2A_0A_3 & x^3+\&c.\\
+\ A_1{}^2 & +2A_1A_2 &
\end{array}$$

Hence, by putting the corresponding coefficients equal, we get

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* To show the identity of form, the first member of this may be written $a+0x+0x^2+\&c.$

† It is besides *only* in this way, that the members can be made universally equal. For if a be transposed to the second member, we shall have an equation of the form, $A'_0+A'_1x+A'_2x^2+\ \&c.=0$: and, by the nature of equations, if A'_0, A'_1, &c., do not each become 0, but have finite values, the equation will be satisfied only when x is taken equal to a root of the equation, and not when it is of any value whatever.

It is plain from what we have thus seen, that if there be two series, $A_0+A_1x+\&c.$, and $B_0+B_1x+\&c.$, which are equal for all values of x, we must have $A_0=B_0$, $A_1=B_1$, &c. For, by putting the two series equal, and by transposition, we get $A_0-B_0+(A_1-B_1)x+\&c.=0$; which cannot be true universally, unless $A_0=B_0$, $A_1=B_1$, &c.

$A_0^2=a^2$, $2A_0A_1=1$, $2A_0A_2+A_1^2=0$, $2A_0A_3+2A_1A_2=0$, &c. The first of these gives $A_0=a$; and by substituting this in the second, transposing, and dividing by $2a$, we find $A_1=\frac{1}{2a}$. By substituting these values in like manner in the third, we readily find $A_2=-\frac{1}{2.4a^3}$; while the next would give, in a similar way, $A_3=\frac{1}{2.2.4a^5}=\frac{1.3}{2.4.6a^5}$: and thus we might proceed without limit. Substituting these values, therefore, in the assumed equation, we get

$$\sqrt{(a^2+x)}=a+\frac{1}{2}.\frac{x}{a}-\frac{1}{2.4}.\frac{x^2}{a^3}+\frac{1.3}{2.4.6}.\frac{x^3}{a^5}-\text{ \&c.}$$

The negative value of $\sqrt{(a^2+x)}$ would be found by changing the sign of a in every term, or, which is the same, by changing the signs of all the terms of the series.

Exercises. Develope the following quantities by means of indeterminate coefficients.

1. $\frac{1}{1-x+x^2}$. *Ans.* $1+x-x^3-x^4+x^6+x^7-x^9-$ &c.; or, $(1+x)(1-x^3+x^6-x^9+\text{ \&c.})$

2. $\frac{1}{1+x+x^2+x^3}$. *Ans.* $1-x+x^4-x^5+x^8-x^9+$ &c.; or, $(1-x)(1+x^4+x^8+x^{12}+\text{ \&c.})$

3. $\sqrt{(1+x^2)}$. *Ans.* $1+\frac{x^2}{2}-\frac{x^4}{2.4}+\frac{1.3x^6}{2.4.6}-\frac{1.3.5x^8}{2.4.6.8}+$ &c.

205. The object of the *binomial theorem* is to determine any power of a binomial. To facilitate the process for establishing this important theorem, we may examine the form of certain expressions that will occur in its investigation. In the first place, then, by taking the product of the two polynomials in the margin, which contain the successive integral powers of x, we find that

$$\begin{array}{l|l|l|l}
\multicolumn{4}{l}{1+p_1x+q_1x^2+r_1x^3+\text{ \&c.}} \\
\multicolumn{4}{l}{1+p_2x+q_2x^2+r_2x^3+\text{ \&c.}} \\
\hline
1+p_1 & x+q_1 & x^2+r_1 & x^3+\text{ \&c.} \\
\quad +p_2 & \quad +p_1p_2 & \quad +p_2q_1 & \quad +\text{ \&c.} \\
 & \quad +q_2 & \quad +p_1q_2 & \quad +\text{ \&c.} \\
 & & \quad +r_2 & \quad +\text{ \&c.} \\
\hline
\multicolumn{4}{l}{1+(p_1+p_2)x+B_2x^2+B_3x^3+\text{ \&c.}}
\end{array}$$

the product is of the form $1+(p_1+p_2)x+B_2x^2+B_3x^3+$ &c., the coefficient of x being the sum of its coefficients in the proposed factors; and the other coefficients, which for brevity are denoted by B_2, B_3, &c., being quantities independent of x. Were we to multiply this product by another similar polynomial $1+p_3x+q_3x^2+$ &c., we should find, in like manner, that the form of the product would be

$$1+(p_1+p_2+p_3)x+C_2x^2+C_3x^3+\text{ \&c.},$$

the coefficient of x being $p_1+p_2+p_3$, and the other coefficients C_2, C_3, &c., being independent of x: and by multiplying by other like polynomials, we should find from the mode in which the successive products are generated, that the first term must always be 1; that the second must be x with a coefficient which is equal to the sum of the coefficients of x in the various factors; and that the remaining terms would be x^2, x^3, &c., with coefficients independent of x; so that the product would be of the form,

$$1+(p_1+p_2+p_3+\ \ldots\ +p_n)x+A_2x^2+A_3x^3+\text{ \&c.}$$

A_2*, A_3, &c., depending only on the coefficients p_1, q_1, ... p_2, q_2, ..., &c.

206. If the polynomial factors have all the same coefficients, p_1, q_1, &c., and if their number be n, we shall evidently have

$$(1+p_1x+q_1x^2+\text{ \&c.})^n=1+np_1x+A_2x^2+A_3x^3+\text{ \&c.}:$$

and hence it appears, that, n being a whole positive number, the first term of the nth power of the polynomial, $1+p_1x+q_1x^2+$ &c., is 1, and the second np_1x; and that the remaining terms consist of the several succeeding powers of x, with coefficients which depend only on p_1, q_1, &c., and on n the index of the power.

207. Hence, we infer conversely, that the nth root of a polynomial of the form, $1+p_1x+p_2x^2+p_3x^3+$ &c., is of the form, $1+\frac{p_1}{n}x+A_2x^2+A_3x^3+$ &c.; since, by the last §, the nth power of this quantity will have its first and second terms the same as those of $1+p_1x+p_2x^2+$ &c., and will have its other terms of the form p_2x^2, p_3x^3, &c. In the nth root, also, of the polynomial, $1+p_1x+p_2x^2+$ &c., x cannot occur with either a fractional or a negative index; as it would be seen by multipli-

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* In what follows, A_2, B_2, &c., are taken in the sense here explained, that is, as merely denoting quantities independent of x, without regard to what, in each particular instance, the values of those quantities may be.

cation, that any power of a quantity such as $1+p_1x+px^{\frac{3}{2}}+$ &c., would necessarily contain terms having fractional indices; and that, in like manner, a power of $1+p_1x+px^{-2}+$ &c., would have terms with negative indices.

208. If in § 206., in which it will be recollected n is a whole positive number, p_1 be taken equal to 1, and q_1, r_1, &c., each equal to 0, we shall have $(1+x)^n=1+nx+A_2x^2+A_3x^3+$ &c., where A_2, A_3, &c., are still independent of x, whatever change their values may have undergone.

If n be a positive fraction, having p for numerator and q for denominator, $(1+x)^n$ will then be the qth root of $(1+x)^p$. But, by what we have just seen, $(1+x)^p=1+px+A_2x^2+A_3x^3+$ &c., and by the last §, we shall have $(1+x)^{\frac{p}{q}}=1+\frac{p}{q}x+A_2x^2+A_3x^3+$ &c.; so that here also the coefficient of the second term is the index of the power, and the coefficients of the succeeding terms are independent of x.

Lastly, if n be negative, we have

$$(1+x)^{-n}=\frac{1}{(1+x)^n}=\frac{1}{1+nx+A_2x^2+A_3x^3+\text{ \&c.}}$$

or, by actual division $(1+x)^{-n}=1-nx+A_2x^2+A_3x^3+$ &c. Hence, we see, that whether the index n is whole or fractional, positive or negative, the first term of the developement of $(1+x)^n$ is 1, and the second nx; and that the remaining terms are x^2, x^3, &c., with coefficients, A_2, A_3, &c., which are independent of x.

209. If we change x into $\frac{x}{a}$, the developement which we have found will become

$$\left(1+\frac{x}{a}\right)^n=1+n\frac{x}{a}+A_2\frac{x^2}{a^2}+A_3\frac{x^3}{a^3}+\text{\&c.};$$

or, (§ 98.) by multiplying by a^n,

$$(a+x)^n=a^n+na^{n-1}x+A_2a^{n-2}x^2+A_3a^{n-3}x^3+\text{\&c.}$$

210. All that now remains to be done for establishing the binomial theorem, is to determine the coefficients A_2, A_3, &c., in the developement just found. To effect this, let a be changed in that developement into 1, and x into $y+z$ or $z+y$; and, again, in the same, let a be changed into $1+y$, and x into z: then, since the coefficients are independent of x, they will undergo no change, and we shall have the two following expressions:

$$1+z+y)^n=$$
$$1+n(z+y)+A_2(z+y)^2+A_3(z+y)^3+\&c. \ldots (1.),$$
and $$(1+y+z)^n=(1+y)^n+n(1+y)^{n-1}z+A_2(1+y)^{n-2}z^2$$
$$+A_3(1+y)^{n-3}z^3+\&c. \ldots \ldots \ldots (2.)$$

Now, the second members of these equations are identical, as they are merely different forms of the same expansion; and therefore (§ 204.) the coefficients of like combinations of y and z contained in them will be equal. The first and second terms of the binomials contained in these members can be found by means of § 208. and § 209. Thus, in (1.) the first and second terms of $(z+y)^2$, $(z+y)^3$, …., $(z+y)^n$, are z^2+2zy, z^3+3z^2y, …., $z^n+nz^{n-1}y$; while, in the second, the like terms of $(1+y)^n$, $(1+y)^{n-1}$, &c., are $1+ny$, $1+(n-1)y$, &c. It is only, however, the second terms, those containing y, which it is necessary to consider for our present purpose; and taking only those terms, we readily get from (1.) the first, and from (2.) the second, of the following lines:

$$ny, \quad 2A_2zy, \quad 3A_3z^2y, \quad 4A_4z^3y, \ \&c.$$
$$ny, \quad n(n-1)zy, \quad A_2(n-2)z^2y, \quad A_3(n-3)z^3y, \ \&c.$$

Hence (§ 204.) we have

$$n=n, \ 2A_2=n(n-1), \ 3A_3=A_2(n-2), \ 4A_4=A_3(n-3), \ \&c.$$

The second of these gives $A_2=\dfrac{n(n-1)}{1.2}$; and substituting this value of A_2 in the third, and dividing by 3, we get $A_3=\dfrac{n(n-1)(n-2)}{1.2.3}$. In a similar manner, we should find $A_4=\dfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}$, and so on.

Substituting, therefore, these values of A_2, A_3, &c., in the developement in § 209., we get

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1.2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3$$
$$+\&c.$$

which is the required formula for the developement of any power of a binomial.*

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* It is easy to see that the general term, T_r is $=n\dfrac{(n-1)(n-2)\ldots.\{n-(r-2)\}}{1.2.3\ldots.(r-1)}a^{n-(r-1)}x^{r-1}$; and also that $T_r=$

211. By dividing each of the terms of the foregoing developement after the first by the term immediately preceding it, we get $n\frac{x}{a}$, $\frac{1}{2}(n-1)\frac{x}{a}$, $\frac{1}{3}(n-2)\frac{x}{a}$, &c.; and therefore, conversely, if the first term be multiplied by the first of these quotients, the product will be the second; if the second be multiplied by the next

It is farther evident, that when n is a whole positive number, the developement, as we should expect from multiplication, will always terminate, having a finite number of terms; as in some one of the coefficients, the factor $n-n$ will arise, and thus the term in which that factor occurs will vanish, as will also all that follow it, and the number of terms will be $n+1$. In every other case, the developement will be an infinite series; as when n is either fractional or negative, none of the factors n, $n-1$, $n-2$, &c., can become nothing. Thus, $(a+x)^7$ will have eight terms and no more; but the developements of $(a+x)^{\frac{3}{4}}$, and $(a+x)^{-4}$ will each consist of an infinite number of terms.

It is easy to show also, that when n is a whole number, the coefficients of terms in the developement of $(a+x)^n$ equally distant from the first and last are equal; and therefore, when the coefficients have been computed up to the middle of the developement, the remaining ones will be had by taking these in a reversed order; and it is plain, that when n is even, there will be a *middle* term differing in its coefficient from all the others.

If a and x be taken each $=1$, we shall have $(1+1)^n$, or $2^n=1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+$ &c. In this the second member is the sum of the coefficients of the binomial; and therefore, to find their sum, we have simply to raise 2 to the nth power. Thus, in Exam. 1. the sum of the coefficients 1, 6, 15, &c., is 2^6, or 64; and the same would be obtained by their actual addition. In like manner, in the expansion of $(a+x)^{\frac{3}{2}}$, the sum of the coefficients will be $2^{\frac{3}{2}}$ or $2\sqrt{2}$, the sum of the positive coefficients exceeding that of the negative by that amount.

If, again, a be taken $=1$, and $x=-1$, we shall have $0^n=1-n+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}+$ &c.; and since the second member consists of the coefficients of $(a-x)^n$, it follows that the sum of these is 0^n, an expression which is nothing, when n is positive, and infinite when it is negative. If $n=0$, the value of 0^n is indeterminate, unless we know that both the binomial and the index vanish simultaneously by giving a certain value to quantity common to both. Thus, if in the expression $(1-x)^{n-nx}$, we take $x=1$, we get 0^0, which can be shown to be $=1$.

$T_{r-1}\frac{n-(r-2)}{r-1}\frac{x}{a}$: whence it follows that the rth term will be derived from the one immediately preceding it by multiplying by $\frac{n-(r-2)}{r-1}\frac{x}{a}$.

quotient, the product will be the third term, and so on. Hence, if the terms be called T_1, T_2, T_3, &c. the formula may be written thus:

$$(a+x)^n=a^n+nT_1\frac{x}{a}+\frac{n-1}{2}T_2\frac{x}{a}+\frac{n-2}{3}T_3\frac{x}{a}+\&c.:$$

a form which is convenient in practice.

212. If $a=1$, the two forms become

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\&c.;\text{ and}$$

$$(1+x)^n=1+nT_1x+\tfrac{1}{2}(n-1)T_2x+\tfrac{1}{3}(n-2)T_3x+\&c.$$

Exam. 1. Required the sixth power of $a+x$.

Here we have $n=6$, and consequently the first term is a^6. The quantities, n, $\frac{1}{2}(n-1)$, &c., are 6, $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{1}{6}$, 0, &c. Then, according to § 211., by multiplying the first term by the first of these, and by multiplying the product by x and dividing by a, we find the second term to be $6a^5x$. From this, again, by multiplying by $\frac{5}{2}$ and x, and by dividing by a, we get $15a^4x^2$, which is the third term. The other terms are obtained similarly; and we find

$$(a+x)^6=a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6.$$

Exam. 2. Required the cube root of the fourth power of $a+x$.

This is the same as to find, in a series, the power of $a+x$, whose index is $\frac{4}{3}$. Now, in cases like the present, in which the indices are fractional, the multipliers, $\frac{1}{2}(n-1)$, $\frac{1}{3}(n-2)$, &c., will take a more convenient form, if we change n into $\frac{p}{q}$, as they will then become respectively $\frac{p-q}{2q}$, $\frac{p-2q}{3q}$, $\frac{p-3q}{4q}$, &c. In the present instance, in which $p=4$, and $q=3$, we readily find those multipliers to be $\frac{1}{6}$, $-\frac{2}{9}$, $-\frac{5}{12}$, $-\frac{8}{15}$, &c.; where the law of continuation is manifest, the numerators and denominators after the second quantity being increased in absolute magnitude from term to term by 3. Then, by employing the formula in § 211., we get

$$(a+x)^{\frac{4}{3}}=a^{\frac{4}{3}}+\frac{4}{3}a^{\frac{1}{3}}x+\frac{4.1}{3.6}\cdot\frac{x^2}{a^{\frac{2}{3}}}-\frac{4.1.2}{3.6.9}\cdot\frac{x^3}{a^{\frac{5}{3}}}+\frac{4.1.2.5}{3.6.9.12}\cdot\frac{x^4}{a^{\frac{8}{3}}}-\&c.;$$

the quantities having negative indices being carried to the denominators.

Exam. 3. Required the developement of

$$\frac{a^3}{(a^4-x^4)^{\frac{3}{4}}},\text{ or } a^3(a^4-x^4)^{-\frac{3}{4}}.$$

The work in this example will perhaps be most easily effected by expanding $(a+x)^{-\frac{3}{4}}$; and then, in the result, changing a into a^4, and x into $-x^4$, and, finally, by multiplying by a^3. Now, since $n=-\frac{3}{4}$, we shall find, by taking, as in the last example, $p=-3$, and $q=4$, that the multipliers, $\frac{1}{2}(n-1)$, $\frac{1}{3}(n-2)$, &c. are $-\frac{7}{8}$, $-\frac{11}{12}$, $-\frac{15}{16}$, $-\frac{19}{20}$, &c.: and hence, by § 211., we shall have $(a+x)^{-\frac{3}{4}}=a^{-\frac{3}{4}}-\frac{3}{4}a^{-\frac{7}{4}}x+\frac{3.7}{4.8}a^{-\frac{11}{4}}x^2-\frac{3.7.11}{4.8.12}a^{-\frac{15}{4}}x^3+\text{\&c.}$ We have next to change a into a^4, and x into $-x^4$. Now, we saw in § 95., that $(x^m)^n=x^{mn}$; and hence the first term of the foregoing developement will become $(a^4)^{-\frac{3}{4}}$, or a^{-3}. In like manner, $a^{-\frac{7}{4}}$ in the next term will be changed into $(a^4)^{-\frac{7}{4}}$, or a^{-7}; and the other powers of a will be modified in a similar manner. Again, by changing x into $-x^4$, in the second term, we shall render that term positive; while the like change in the next term will simply make x^2 become x^8. By the same substitution in the next term, we shall change the sign, and instead of x^3, we shall have x^{12}. In a similar way, the corresponding changes in the other terms will be made without difficulty; and we shall have

$$(a^4-x^4)^{-\frac{3}{4}}=a^{-3}+\frac{3}{4}a^{-7}x^4+\frac{3.7}{4.8}a^{-11}x^8+\frac{3.7.11}{4.8.12}a^{-15}x^{12}+\text{\&c.}$$

Lastly, by multiplying the result last obtained by a^3, and by taking the quantities with negative indices to the denominators, we get for the required developement,

$$\frac{a^3}{(a^4-x^4)^{\frac{3}{4}}}=1+\frac{3x^4}{4a^4}+\frac{3.7x^8}{4.8a^8}+\frac{3.7.11x^{12}}{4.8.12a^{12}}+\frac{3.7.11.15x^{16}}{4.8.12.16a^{16}}+\text{\&c.}$$

Exercises. Expand the following quantities by means of the binomial theorem.

1. $(a+x)^5$. *Ans.* $a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$.
2. $(a^2-x^2)^5$. *Ans.* $a^{10}-5a^8x^2+10a^6x^4-10a^4x^6+5a^2x^8-x^{10}$.

3. $\sqrt{(a+x)}$. *Ans.* $a^{\frac{1}{2}}\left(1+\frac{x}{2a}-\frac{x^2}{2.4a^2}+\frac{1.3x^3}{2.4.6a^3}-\&c.\right)$

4. $\sqrt{(a-x)}$. *Ans.* $a^{\frac{1}{2}}\left(1-\frac{x}{2a}-\frac{x^2}{2.4a^2}-\frac{1.3x^3}{2.4.6a^3}-\&c.\right)$

5. $(a+x)^{\frac{1}{3}}$. *Ans.* $a^{\frac{1}{3}}\left(1+\frac{x}{3a}-\frac{2x^2}{3.6a^2}+\frac{2.5x^3}{3.6.9a^3}-\&c.\right)$

6. $(a^3+x^3)^{\frac{1}{3}}$. *Ans.* $a\left(1+\frac{x^3}{3a^3}-\frac{2x^6}{3.6a^6}+\frac{2.5x^9}{3.6.9a^9}-\&c.\right)$

7. $(a^3-x^3)^{\frac{1}{3}}$. *Ans.* $a\left(1-\frac{x^3}{3a^3}-\frac{2x^6}{3.6a^6}-\frac{2.5x^9}{3.6.9a^9}-\&c.\right)$

8. $\frac{1}{a+x}$. *Ans.* $\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\&c.$

9. $\frac{1}{(a+x)^2}$. *Ans.* $\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{4x^3}{a^5}+\&c.$

10. $\frac{a^3}{(a^3-x^3)^{\frac{2}{3}}}$. *Ans.* $a+\frac{2x^3}{3a^2}+\frac{2.5x^6}{3.6a^5}+\frac{2.5.8x^9}{3.6.9a^8}+\&c.$

11. $\frac{a-x}{(a+x)^{\frac{1}{3}}}$. *Ans.* $\frac{a-x}{a^{\frac{1}{3}}}\left(1-\frac{x}{3a}+\frac{1.4x^2}{3.6a^2}-\frac{1.4.7x^3}{3.6.9a^3}+\&c.\right)$

CHAPTER XII.

CONTINUED FRACTIONS.

213. An expression of the form $a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\&c.,}}}$ where a_1, a_2, &c., are integers, is called a *continued fraction.* The numerators are not necessarily each equal to unity, as here; but in general it is only fractions such as the one here exhibited, that are of use. For understanding the nature and origin of such expressions, let us consider, as a simple numerical example, the

fraction $\frac{3597}{1558}$. By dividing the numerator by the denominator, we get $2+\frac{481}{1558}$. If the numerator and denominator of the fractional part of this be divided by 481, there results $\frac{1}{3+\frac{115}{481}}$, or as it may be written $\frac{1}{3}{}_{+\frac{115}{481}}$; and consequently we have the proposed fraction $=2+\frac{1}{3}{}_{+\frac{115}{481}}$. Again, by dividing the terms of the last fraction by 115, we find $\frac{115}{481}=\frac{1}{4}{}_{+\frac{21}{115}}$. By a similar process we transform $\frac{21}{115}$ into $\frac{1}{5}{}_{+\frac{10}{21}}$, and $\frac{10}{21}$ into $\frac{1}{2}{}_{+\frac{1}{10}}$; and here the process terminates, as, the numerator of the last fraction being 1, no change would be produced by dividing by it. Hence, by successive substitutions, we get $\frac{3597}{1558}=2+\frac{481}{1558}=2+\frac{1}{3}{}_{+\frac{115}{481}}$

$$=2+\frac{1}{3}{}_{+\frac{1}{4}}{}_{+\frac{21}{115}}=2+\frac{1}{3}{}_{+\frac{1}{4}}{}_{+\frac{1}{5}}{}_{+\frac{10}{21}}=2+\frac{1}{3}{}_{+\frac{1}{4}}{}_{+\frac{1}{5}}{}_{+\frac{1}{2}}{}_{+\frac{1}{10}}.$$

214. By considering the first of the foregoing expressions, we find that the given fraction is greater than 2 and less than 3: and therefore 2 may be considered a first approximation to its value. In the next expression, the denominator $3+\frac{115}{481}$ lying between 3 and 4, the value of $\frac{3597}{1558}$ must lie between $2+\frac{1}{3}$ and $2+\frac{1}{4}$: and therefore $2+\frac{1}{3}$ or $\frac{7}{3}$, the second expression without the fraction $\frac{115}{481}$, is a second approximation to the value of the proposed fraction. The third expression shows that the denominator which, in the last approximation, we took as 3, lies between $3+\frac{1}{4}$ and $3+\frac{1}{5}$: and therefore for another approximation, we shall have $2+\frac{1}{3\frac{1}{4}}$, or $2+\frac{4}{13}$, or finally $\frac{30}{13}$. By using the next fraction $\frac{1}{5}$, and after that $\frac{1}{2}$, we should have a fourth and a fifth approximation; and lastly, by employing $\frac{1}{10}$, we should get $\frac{3597}{1558}$, the original fraction.

215. By reviewing the process in § 213. we shall find it to be exactly the same as that which is employed (§ 81. and *Arithmetic*, p. 81.) in determining the greatest common measure of the numerator and denominator. Thus, we divide 3597 by 1558, and find for remainder 481. We then divide 1558 by 481; and 481 by 115, the next remainder: and thus we proceed till there is no remainder. Then the quotients after the first are the denominators of the several fractions in the continued one.

This process, by following out the same principles that were employed in § 213., may be generalised in the following manner.

$$\frac{p}{q}=a_1+\frac{r_1}{q}=a_1+\cfrac{1}{a_2+\cfrac{r_2}{r_1}}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{r_3}{r_2}+\&c.}}$$

Here, a_1 is put to denote the quotient, and r_1 the remainder obtained in dividing p by q; and it is plain, that if p be less than q, a_1 will be $=0$. Then a_2 is assumed to denote the quotient, and r_2 the remainder, obtained by dividing q by r_1; while a_3 and r_3 are obtained by dividing r_1 by r_2: and thus we may proceed till we have no remainder, or till we have got a sufficient number of terms. This process gives a rèsult exactly of the form mentioned in § 213., and it is evidently the same as that which is employed in finding the greatest common measure of p and q, the numerator and denominator of the original fraction, each divisor being divided by the succeeding remainder. It is plain also, that when p and q are integers, there will at length be obtained a quotient without remainder, since each remainder is an integer, and is less than the one preceding it.

216. From the views which we have had, we can readily find the simple fractions approximating to the value of a continued one. Thus, let the fraction be

$$a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\&c.}}};$$

and let p_1 and q_1 be the numerator and denominator of the first approximate fraction; p_2 and q_2 those of the second, &c. Then

$$\frac{p_1}{q_1}=a_1=\frac{a_1}{1}, \text{ so that } p_1=a_1 \text{ and } q_1=1: \text{ and}$$

$$\frac{p_2}{q_2}=a_1+\frac{1}{a_2}=\frac{a_1a_2+1}{a_2}=\frac{p_1a_2+1}{a_2}.$$

To get the next approximate value, we have merely to change

$$a_2 \text{ into } a_2+\frac{1}{a_3}, \text{ or } a_2+a_3^{-1} \text{ in the last: then,}$$

$$\frac{p_3}{q_3}=\frac{a_1a_2+1+a_1a_3^{-1}}{a_2+a_3^{-1}};$$

whence, by multiplying the numerator and denominator by a_3, and then substituting p_1, q_1, p_2, and q_2 for their equals a_1, 1, a_1a_2+1, and a_2, we get

$$\frac{p_3}{q_3}=\frac{(a_1a_2+1)a_3+a_1}{a_2a_3+1}=\frac{p_2a_3+p_1}{q_2a_3+q_1}.$$

To find the next approximate value, we must change, in the last, a_3 into $a_3+a_4^{-1}$. Then by multiplying by a_4, and substituting p_3 and q_3 for $p_2a_3+p_1$, and $q_2a_3+q_1$, we get

$$\frac{p_4}{q_4}=\frac{p_2a_3+p_1+p_2a_4^{-1}}{q_2a_3+q_1+q_2a_4^{-1}}=\frac{(p_2a_3+p_1)a_4+p_2}{(q_2a_3+q_1)a_4+q_2}=\frac{p_3a_4+p_2}{q_3a_4+q_2}.$$

In a similar manner, we should find $\frac{p_5}{q_5}=\frac{p_4a_5+p_3}{q_4a_5+q_3}$; and in general $\frac{p_r}{q_r}=\frac{p_{r-1}a_r+p_{r-2}}{q_{r-1}a_r+q_{r-2}}$. Hence the law of formation of the several approximate fractions is obvious, and may be thus exhibited:

$$\begin{array}{llllll}
a_1, & a_2, & a_3, & a_4, & a_5, & \&c. \\
a_1, & p_1a_2+1, & p_2a_3+p_1, & p_3a_4+p_2, & p_4a_5+p_3, & \&c. \\
1, & a_2, & q_2a_3+q_1, & q_3a_4+q_2, & q_4a_5+q_3, & \&c.
\end{array}$$

Here the quotients, a_1, a_2, &c., are written in succession, and below them are placed, in one line, the values of p_1, p_2, &c., and, in another, those of q_1, q_2, &c. Then, by what we have seen, the values under a_3 are found by multiplying those under a_2 (p_2 and q_2) by a_3, and to the products adding the corresponding values (p_1 and q_1) under a_1. In exactly the same manner the values under a_4 are derived from those under a_3 and a_2; and universally each pair of values, except those under a_1 and a_2, are obtained by multiplying those last found by the corresponding quotient, and adding severally to the results the two preceding values. The values under a_1 are a_1 and 1; and those under a_2 are found from the last by multiplying by a_2, and adding 1 to the first product.

To exemplify this, let us resume the continued fraction which we obtained in § 213. In it a_1, a_2, &c., were 2, 3, 4, 5, 2, and 10; and the operation for finding the successive approximate values, or *converging fractions* as they have also been called, will be as in the margin. The first fraction is the first quotient with 1 as denominator. To find the second, we multiply the terms of the first by the next quotient 3, adding 1 in the numerator, and the result is $\frac{7}{3}$. We then multiply the 7 and the 3 by the next quotient 4, adding to the products 2 and 1, the terms of the preceding fraction. The rest of the fractions are found

$$\begin{array}{cccccc}
2, & 3, & 4, & 5, & 2, & 10, \\
\frac{2}{1}, & \frac{7}{3}, & \frac{30}{13}, & \frac{157}{68}, & \frac{344}{149}, & \frac{3597}{1558}
\end{array}$$

similarly, and the last of them is $\frac{3597}{1558}$, the original fraction in § 213.

217. If the first of two consecutive converging fractions be taken from the second, the numerator of the remainder is 1 or -1. To prove this, let p and p' be the numerators, and q and q' the denominators, of the two fractions, and let a_r be the next quotient. Then, forming the next converging fraction in the manner shown in § 216., we shall have the three successive fractions $\frac{p}{q}$, $\frac{p'}{q'}$, and $\frac{p'a_r+p}{q'a_r+q}$. Now, if the second of these be taken from the first, and the third from the second, according to § 87., the remainders are $\frac{pq'-p'q}{qq'}$, and $\frac{p'q-pq'}{q'(q'a_r+q)}$, the numerators being the same, but with contrary signs: and, since the fractions that we have employed are any three consecutive ones whatever, it follows that in any number of consecutive fractions, if the second be taken from the first, the third from the second, the fourth from the third, &c., the numerators of the differences will be the same in magnitude, but alternately positive and negative. To find what these numerators are, we have only to take the difference of any two successive ones in those found in the last §. By subtracting, therefore, the second from the first, we get $\frac{a_1}{1}-\frac{a_1a_2+1}{a_2}=\frac{-1}{a_2}$: and the numerator of this difference being -1, it follows from what we have seen that the next difference will have 1 as numerator, the next to that -1, &c. This may be illustrated in numbers by taking, in the last example, $\frac{7}{3}$ from $\frac{2}{1}$, $\frac{30}{13}$ from $\frac{7}{3}$, or, in general, any of the converging fractions from the one immediately preceding it.

218. It is an important property of the converging fractions, that any two consecutive ones, except the last two, when the fraction terminates, are one of them greater and the other less than the value of the continued fraction from which they arise. To prove this, let the value of the continued fraction be denoted by x, and let the successive fractions be as in the last §. Then, if we take a to denote $a_r+\frac{1}{a_{r+1}+\&c.}$, the part of the continued fraction commencing with a_r, we shall get the exact value of the whole continued fraction by changing a_r into a in the third

fraction: that is, we shall have $x=\frac{p'\alpha+p}{q'\alpha+q}$. By subtracting from this, successively, $\frac{p}{q}$ and $\frac{p'}{q'}$, we get, after easy modifications, $x-\frac{p}{q}=\frac{p'q-pq'}{q'\alpha+q}\cdot\frac{\alpha}{q}$, and $x-\frac{p'}{q'}=\frac{pq'-p'q}{(q'\alpha+q)q'}=-\frac{p'q-pq'}{q'\alpha+q}\cdot\frac{1}{q'}$. Now, these have evidently opposite signs; and therefore, if x be greater than the one fraction pq^{-1}, it must be less than the other $p'q'^{-1}$, and if it be less than the former, it must be greater than the latter: and if we consider the first and second terms a_1 and $a_1+a_2{}^{-1}$ in § 215., we shall see that x is greater than a_1 by what follows that fraction; but that it is less than the second, because the denominator a_2 is too small by what follows it, and the denominator being too small the fraction is too great. Hence x is greater than the first of the converging fractions, and less than the second; and being less than the second it must, by what we have seen, be greater than the third, and so on. It thus appears, that, with the exception of the last, when the fraction is finite, the converging fractions occupying the first place, the third, the fifth, &c., are too small in value, and that the others are too great. Thus, in the foregoing example, 2, $\frac{30}{13}$, and $\frac{344}{149}$ are too small, and $\frac{7}{3}$ and $\frac{157}{68}$ too great.

219. By considering the differences obtained in the last §, between x and two consecutive converging fractions, we find that the latter fraction differs less from x than the former one does; since, of the multipliers, $\frac{\alpha}{q}$ and $\frac{1}{q'}$, in which alone, exclusive of their signs, they differ, the latter is less than the former, both because q' is greater than q, and 1 less than α, as is plain from considering the nature and formation of the successive fractions. Hence we arrive at the important conclusion, that each converging fraction is nearer the value of the continued fraction, than the preceding converging one; and hence it is, that these fractions are so called.

220. We saw in § 217. that the difference of any two consecutive converging fractions, $\frac{p}{q}$ and $\frac{p'}{q'}$, is $\frac{1}{qq'}$: and, by the last §, this is greater than the difference between either of these fractions and the continued one. Hence, if we take any converging fraction as

the value of the continued one, the error will be less than 1 divided by the product of the denominator of that converging fraction into the denominator of the one after it: and thus we have a simple and convenient mode of estimating the degree of approximation attained at any particular step in the process.

221. Let $\frac{p}{q}$, $\frac{p'}{q'}$, and $\frac{p''}{q''}$ be three fractions, of which the first and second are successive converging ones. Then, by § 217. and by subtraction, we have $\frac{p}{q}-\frac{p'}{q'}=\frac{\pm 1}{qq'}$, and $\frac{p}{q}-\frac{p''}{q''}=\frac{pq''-p''q}{qq''}$. Now, the numerator of the second of these differences being evidently a whole number, cannot be less in absolute value than 1, the numerator of the first, and its denominator will be less than qq', the denominator of the other, if q'' be less than q'. Whenever, therefore, q'' is less than q', the last of the three fractions will differ more from the first than the second will; and will consequently be farther from the value of the continued fraction than the second differs from it, since (§ 218.) F lies between the first and second.

222. The principles that have been established regarding continued fractions, enable us to approximate to the ratios of numbers that are inconveniently large, and of which in consequence the mind feels difficulty in estimating the comparative magnitudes. Thus, if the fraction $\frac{24217}{40620}$ be proposed, its terms are so large, that we have a very inadequate idea of its magnitude. Let us, therefore, convert it (§ 215.) into a continued fraction; or, what is sufficient, let us determine the several denominators of the fractions forming the continued one. To do this, we divide the numerator by the denominator, which gives nothing for the first quotient. We then divide the denominator by the numerator; the numerator by 16403, the remainder; this remainder by 7814, the next remainder; and so on, as in finding the greatest common measure. By going through with the full work, we find the quotients to be 0, 1, 1, 2, 10, 12, 9, and 7. Then (§ 216.) the process for finding the converging fractions will be as follows:

$$\begin{array}{cccccccc} & 1, & 1, & 2, & 10, & 12, & 9, & 7. \\ \frac{0}{1}, & \frac{1}{1}, & \frac{1}{2}, & \frac{3}{5}, & \frac{31}{52}, & \frac{375}{629}, & \frac{3406}{5713}, & \frac{24217}{40620}. \end{array}$$

From these, omitting the first three expressions, as being evidently far from the truth, we see that the value of the proposed fraction is nearly $\frac{3}{5}$, more nearly $\frac{31}{52}$, more nearly still $\frac{375}{629}$, and still much more nearly $\frac{3406}{5713}$; while, as it ought to be, the last fraction is the same as the proposed one.

As to the degree of approximation, $\frac{3}{5}$ must differ (§ 220.) from the truth by less than $\frac{1}{260}$, 260 being the product of the denominators 5 and 52; and $\frac{31}{52}$ must be in error by less than $\frac{1}{32708}$, 32708 being the product of 52 and 629. The limit of the error with regard to the next two fractions would be found in a similar way.

223. As another example, let it be required to find fractions approximating to the ratio of the circumference of a circle to its diameter, which (*Diff. and Int. Calc.*, p. 41.) is greater than $\frac{31415926535}{10000000000}$ and less than $\frac{31415926536}{10000000000}$. By employing, with regard to these two fractions, the usual process for getting the greatest common measure we find, that they both give the first seven quotients, 3, 7, 15, 1, 292, 1, 1, the same; while the others would be different, and are therefore not to be used. Then the converging fractions are found in the following manner:*

$$\begin{array}{ccccccc} 3, & 7, & 15, & 1, & 292, & 1, & 1, \\ \frac{3}{1}, & \frac{22}{7}, & \frac{333}{106}, & \frac{355}{113}, & \frac{103993}{33102}, & \frac{104348}{33215}, & \frac{208341}{66317}. \end{array}$$

224. As a third example, let it be required to express the square root of 13 by a continued fraction, and to find the ordinary fractions converging its value.

Here the square root of 13 being between 3 and 4, we get equation (1.), where x_1 must plainly be greater than 1. From this, by transposing 3 and taking the reciprocals, we get the first equality in line (2.). The next expression in that line is got by

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* Two of these, $\frac{22}{7}$ and $\frac{355}{113}$, are more useful in practice than the others. The first was discovered by Archimedes; and, on account of the smallness of its terms, it is convenient in practice. It is deficient in accuracy, however, giving the circumference too great by rather more than 1, when the diameter is 800. The second was given by Metius, a Dutch mathematician of the seventeenth century. This is an exceedingly close approximation, making the circumference too great by very little more than 1, when the diameter is 4,000,000: and it is easily recollected, the numbers composing its denominator and numerator (113,355) being the first three odd numbers each taken twice.

In the solution of this interesting question, Bonnycastle, Bourdon, and some other writers on algebra, have fallen into error in not having taken two limits, and in having carried out the fraction no farther than $\frac{314159}{100000}$. In this way they obtain the converging fractions, $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\frac{9208}{2931}$, $\frac{9563}{3044}$, &c.; only the first four of which are correct. Lagrange, in his *Additions to Euler's Algebra*, has given the solution with great fullness and accuracy, and has determined the first thirty-four converging fractions.

$$\sqrt{13}=3+\frac{1}{x_1} \ldots\ldots\ldots\ldots (1.)$$

$$x_1=\frac{1}{\sqrt{13}-3}=\frac{\sqrt{13}+3}{4}=1+\frac{1}{x_2} \ldots (2.)$$

$$x_2=\frac{4}{\sqrt{13}-1}=\frac{\sqrt{13}+1}{3}=1+\frac{1}{x_3} \ldots (3.)$$

$$x_3=\frac{3}{\sqrt{13}-2}=\frac{\sqrt{13}+2}{3}=1+\frac{1}{x_4} \ldots (4.)$$

$$x_4=\frac{3}{\sqrt{13}-1}=\frac{\sqrt{13}+1}{4}=1+\frac{1}{x_5} \ldots (5.)$$

$$x_5=\frac{4}{\sqrt{13}-3}=\sqrt{13}+3=6+\frac{1}{x_6} \ldots (6.)$$

$$x_6=\frac{1}{\sqrt{13}-3}=x_1 \ldots\ldots\ldots\ldots (7.)$$

Therefore,

$$\sqrt{13}=3+\frac{1}{1}{}_{+}\frac{1}{1}{}_{+}\frac{1}{1}{}_{+}\frac{1}{1}{}_{+}\frac{1}{6}{}_{+}\frac{1}{1}{}_{+\&c.}$$

multiplying the numerator and denominator of the preceding fraction by $\sqrt{13}+3$, according to § 108. Then, $\sqrt{13}+3$ being between 6 and 7, we get the concluding expression by dividing by 4: and x_2 must also be greater than 1. The first equality in line (3.) is found by transposing 1 in the concluding part of (2.), and taking the reciprocals; and, after this, the work throughout proceeds just as in line (2.). In line (7.) we get for x_6 the same value that in line (2.) we found for x_1; and therefore it is unnecessary to proceed farther, as x_7, x_8, &c. would plainly be the same as x_2, x_3, &c. Hence the continued fraction is *periodic*, as the numbers 1, 1, 1, 1, 6, commencing in equation (2.) and ending in (6.), would evidently recur perpetually. The converging fractions are found as follows, in the usual manner.

$$\begin{array}{cccccccccc} 3, & 1, & 1, & 1, & 1, & 6, & 1, & 1, & 1, & \&c. \\ \frac{3}{1}, & \frac{4}{1}, & \frac{7}{2}, & \frac{11}{3}, & \frac{18}{5}, & \frac{119}{33}, & \frac{137}{38}, & \frac{256}{71}, & \frac{393}{109}, & \&c. \end{array}$$

225. In a manner exactly similar to that which was employed in the last §, the roots of quadratic equations may be expressed in continued fractions. Thus, if the equation $3x^2-3x-1=0$ be proposed, we have (§ 151.)

$$x=\frac{3\pm\sqrt{21}}{6}; \text{ that is } x=\frac{\sqrt{21}+3}{6}, \text{ and } x=-\frac{\sqrt{21}-3}{6}.$$

The work for obtaining the continued fraction equivalent to the positive root is as follows.

$$\frac{\sqrt{21}+3}{6}=1+\frac{1}{x_1}\ \dots\dots\dots\dots\ (1.)$$

$$x_1=\frac{6}{\sqrt{21}-3}=\frac{\sqrt{21}+3}{2}=3+\frac{1}{x_2}.\ (2.) \qquad x=1+\frac{1}{3+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+\&c.}$$

$$x_2=\frac{2}{\sqrt{21}-3}=\frac{\sqrt{21}+3}{6}=1+\frac{1}{x_3}.\ (3.) \qquad x=-\frac{1}{3+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+\&c.}$$

$$x_3=\frac{6}{\sqrt{21}-3}=x_1\ \dots\dots\dots\dots\ (4.)$$

The finding of the negative root proceeds in exactly the same manner, except that we have at first $\frac{\sqrt{21}-3}{6}=0+\frac{1}{x_1}$. The two continued fractions are $1+\frac{1}{3+\&c.}$, and $-\frac{1}{3+\&c.}$, as exhibited above.

The converging fractions would be found in the usual way to be $\frac{1}{1}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{19}{15}$, &c., and $-\frac{1}{3}$, $-\frac{1}{4}$, $-\frac{4}{15}$, &c.

226. Whenever the square root is to be taken, as in the last two examples, the continued fraction is periodic; but the limits of the present publication do not admit the proof of this. It is easy to show, that, conversely, *any periodic continued fraction is one of the roots of a quadratic equation.* As an example, let us take the fraction, $x=2+\frac{1}{3+}\frac{1}{1+\&c.}$, the denominators 3 and 1 being continually repeated. By transposing 2, we find that $x-2$ is the value of all the periodical part. Annexing this, therefore, at the end of the first period, we get

$$x-2=\frac{1}{3+}\frac{1}{1+x-2}, \qquad \text{or } x-2=\frac{1}{3+}\frac{1}{x-1}.$$

Then, the converging fractions will be found in the usual manner, as in the margin; and as the denominators are finite in number, the last of these converging fractions is exact. Putting it, therefore, equal to $x-2$, multiplying by its denominator $3x-2$, and transposing, we get $3x^2-9x+5=0$, the equation required.

$$\begin{array}{ccc} 0, & 3, & x-1; \\ \frac{0}{1}, & \frac{1}{3}, & \frac{x-1}{3x-2}. \end{array}$$

The other root of this equation would have for quotients 0, 1, 2, 1, 3, 1, 3, 1, 3, &c. As the reduction of the continued fraction given by these to an equation is a little more difficult than that of the foregoing, we may employ the following method, which is applicable in all cases.

$$\text{Let } x=0+\frac{1}{1+}\frac{1}{2+}\frac{1}{y}; \text{ where } y=1+\frac{1}{3+}\frac{1}{y}$$

as would be evident by carrying the fraction out for a few terms. Then, by the usual process, as in the margin, we find $x=\frac{2y+1}{3y+1}$, and $y=\frac{4y+1}{3y+1}$. Lastly, by finding the value of y in terms of x in the first of these, and substituting it in the second, we get, after some reductions, $3x^2-9x+5=0$, as before.

$0,\ 1,\ 2,\ y;$

$\frac{0}{1},\ \frac{1}{1},\ \frac{2}{3},\ \frac{2y+1}{3y+1}.$

$1,\ 3,\ y;$

$\frac{1}{1},\ \frac{4}{3},\ \frac{4y+1}{3y+1}.$

Exercises. Find the successive quotients, and the converging fractions, belonging to the following quantities.

Given.		Quotients.	Converging Fractions.
1. $\frac{351}{965}$.	*Ans.*	0, 2, 1, 2, 1, 87.	$\frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}$.
2. $\frac{764}{251}$.		3, 22, 1, 4, 2.	$\frac{3}{1}, \frac{67}{22}, \frac{70}{23}, \frac{347}{114}$.
3. $\frac{5637}{1769}$.		3, 7, 1, 2, 4, 5, 1, 2.	$\frac{3}{1}, \frac{22}{7}, \frac{25}{8}, \frac{72}{23}, \frac{313}{100}, \frac{1637}{523}, \frac{1950}{623}$.
4. $\frac{196}{121}$.		1, 1, 1, 1, 1, 1, 2, 2, 2.	$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{34}{21}, \frac{81}{50}$.
5. $\sqrt{2}$.		1, 2, 2, 2, &c.	$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}$, &c.
6. $\sqrt{28}$.		5, 3′, 2, 3, 10′, &c.*	$\frac{5}{1}, \frac{16}{3}, \frac{37}{7}, \frac{127}{24}, \frac{1307}{247}$, &c.
7. $\sqrt{45}$.		6, 1′ 2, 2, 2, 1, 12′, &c.	$\frac{6}{1}, \frac{7}{1}, \frac{20}{3}, \frac{47}{7}, \frac{114}{17}, \frac{161}{24}, \frac{2046}{305}$, &c.
8. $\sqrt{52}$.		7, 4′, 1, 2, 1, 4, 14′, &c.	$\frac{7}{1}, \frac{29}{4}, \frac{36}{5}, \frac{101}{14}, \frac{137}{19}, \frac{649}{90}, \frac{9223}{1279}$, &c.
9. $\sqrt{53}$.		7, 3′, 1, 1, 3, 14′, &c.	$\frac{7}{1}, \frac{22}{3}, \frac{29}{4}, \frac{51}{7}, \frac{182}{25}, \frac{2599}{357}$, &c.

10. Find the fractions converging to the ratio of one day (86400 seconds) to 5 hours 48 min. 49 sec. (20929 seconds), the excess of the solar or civil year above 365 days.

Ans. $\frac{4}{1}, \frac{29}{7}, \frac{33}{8}, \frac{128}{31}, \frac{161}{39}, \frac{2704}{655}, \frac{2865}{694}, \frac{5569}{1349}$.

11. Find the fractions converging to the ratio of the year, as in the last exercise, to 29 days 12 hours 44 min. 3 sec., the

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* In this and other instances, accents are put over the first and last numbers of the period.

mean synodical lunar month (the time between one new or full moon and the following).

Ans. $\frac{12}{1}, \frac{25}{2}, \frac{37}{3}, \frac{62}{5}, \frac{99}{8}, \frac{1349}{109}, \frac{1448}{117}, \frac{2797}{226}, \frac{267163}{21587}, \frac{269960}{21813}$.

12. Find the equation which has, for one of its roots, 3 together with the continued fraction having for denominators 2, 1, 4′, 2, 3′, &c. *Ans.* $121x^2-845x+1473=0$.

CHAPTER XIII.

INDETERMINATE ANALYSIS.

227. We saw (§ 142.) that problems are unlimited, admitting of an infinite number of answers, when there are fewer equations given than there are unknown quantities to be determined. In many instances, however, the answers are to be whole numbers; and this limitation, excluding all fractional results, greatly diminishes the number of answers. If, also, the answers are to be positive, which is always the case, when they are to agree literally with the enunciation of the problem in its plain ordinary meaning, their number is still farther limited by this restriction; and it may even happen, in either case, that the question will admit of no solution whatever.*

As an example of a problem of this kind, let it be required to find how many five-pound notes and guineas must be taken to pay a bill of £512. Here, the number of shillings in five pounds, in a guinea, and in £512, being respectively 100, 21, and 10240, if we put x and y to denote the number of five-pound notes and the number of guineas, we shall have $100x+21y=10240$; and from the nature of the question, according to its meaning in common language, x and y are to be whole positive numbers, since no fractional parts of either five-pound notes or guineas are admissible. In the operation in the margin, equation (2.) is obtained from (1.) by resolving it for y*; and (3.) is the same as (2.), the actual division of 10240 and $100x$ by 21 being performed. Now, the second member of (3.) being the value of y, must be an integral number, and its first and second terms being integers already, its third must be so likewise. We put this term, therefore, equal to v_1, and thus obtain (4.); from which, by resolving it for x, we get (5.): and (5) is changed into (6.) by actually dividing $-21v_1$ by 16. Now, since x is to be an integer, its two terms in (6.) must give an integer; but the second of them being already an integer, the first must be so likewise: we therefore put it equal to v_2, which

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* No equation of the form $acx \pm bcy = d$, that is, no equation in which the coefficients of x and y have an integral common measure c, will admit of a solution in whole numbers, unless d be also divisible by c: for if the equation be put under the form $c(ax+by)=d$, it is plain that d, being the product of c and $ax+by$, must have c as factor. In numerous other instances also, when a and b are positive, the equation $ax+by=c$ admits of no solution in whole positive numbers. Thus, for example, no whole positive values of x and y will satisfy the equation $5x+7y=4$.

$$100x+21y=10240 \;\ldots\ldots\; (1.)$$

$$y=\frac{10240-100x}{21} \;\ldots\ldots\; (2.)$$

$$y=487-4x+\frac{13-16x}{21}\dagger \;\ldots\; (3.)$$

$$\frac{13-16x}{21}=v_1 \;\ldots\ldots\; (4.)$$

$$x=\frac{13-21v_1}{16} \;\ldots\ldots\; (5.)$$

$$x=\frac{13-5v_1}{16}-v_1 \;\ldots\ldots\; (6.)$$

$$\frac{13-5v_1}{16}=v_2 \;\ldots\ldots\; (7.)$$

$$v_1=\frac{13-16v_2}{5} \;\ldots\ldots\; (8.)$$

$$v_1=2-3v_2+\frac{3-v_2}{5} \;\ldots\ldots\; (9.)$$

$$\frac{3-v_2}{5}=v \;\ldots\ldots\; (10.)$$

$$v_2=3-5v \;\ldots\ldots\; (11.)$$

$$v_1=16v-7 \;\ldots\ldots\; (12.)$$

$$x=10-21v \;\ldots\ldots\; (13.)$$

$$y=440+100v \;\ldots\ldots\; (14.)$$

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* The equation should be resolved for the quantity which has the less coefficient, as by this means the work is commonly shortened.

† Though, for the sake of uniformity of process, which is desirable in a first example, 21 has been taken as contained only 487 times in 10240, yet it is contained *more nearly* 488 times, there being a defect of only 8, while in (3.) in the text there is an excess of 13. In like manner, 21 is contained in 100 much more nearly 5 times than 4 times, the one difference being only 5, while the other is 16. 100 in fact is taken as being $105-5$. Proceeding on this principle, we shall have the remainder of the operation as in the margin. For the same reason, in getting (6.), 5 is taken as being contained twice in 8 with the defect 2; or, what amounts to the same, 8 is taken as being $10-2$. The values of x and y in (9.) and

$$y=488-5x+\frac{5x-8}{21} \;\ldots\; (3.)$$

$$\frac{5x-8}{21}=v_1 \;\ldots\ldots\; (4.)$$

$$x=\frac{21v_1+8}{5} \;\ldots\ldots\; (5.)$$

$$x=4v_1+2+\frac{v_1-2}{5} \;\ldots\; (6.)$$

$$\frac{v_1-2}{5}=v \;\ldots\ldots\; (7.)$$

$$v_1=5v+2 \;\ldots\ldots\; (8.)$$

$$x=21v+10 \;\ldots\ldots\; (9.)$$

$$y=440-100v \;\ldots\ldots\; (10.)$$

gives (7.). By processes exactly similar, we get equations (8.), (9.), (10.), and (11.): and as the value of v_2 in the last of these is integral, we proceed no farther on this principle. We have now, by retracing our steps, to find the values of v_1, x, and y in terms of v. Thus, (12.) is found by substituting in (9.), v and $3(3-5v)$, for what are shown in (10.) and (11.) to be their equals. Then (13.) is obtained from (6.) in a similar manner, by means of (7.), (11.), and (12.): and lastly, (14.) is derived from (3.) by means of (13.), (4.), and (12.).

The values of x and y thus found in (13.) and (14.) will evidently be whole numbers, whatever integral values are given to v. Since, however, x and y must be positive, it is plain that v must not be taken positive, as any such value (1, 2, &c.) would give negative values (-11, -32, &c.) for x. Let us, therefore, take v successively equal to 0, -1, -2, &c., and we shall obtain the annexed system of values for x and y, which will all be found to answer; so that the bill may be paid with 10 five-pound notes and 440 guineas; with 31 of the first and 340 of the second; with 52 and 240; with 73 and 140; or with 94 and 40. Thus, for instance, if we take the third answer, the value of 52 five-pound notes is £260, and that of 240 guineas £252; and £240 + £252 = £512. Had we taken v equal to -5, -6, or any other negative whole number, except -1, -2, -3, and -4, the ones which were employed above, we should have got negative values for y: and therefore the question admits of five solutions, and no more.

v	0,	-1,	-2,	-3,	-4.
x	10,	31,	52,	73,	94.
y	440,	340,	240,	140,	40.

228. By considering the mode of formation of the values of x and y in the foregoing example, it will be seen that each value of x, after the first, is derived from the one immediately pre-

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(10.) are the same as those in (13.) and (14.) in the text, except that the sign of v is different. Hence, in using (9.) and (10.), we take v equal to 1, 2, 3, &c., instead of -1, -2, -3, &c.

In all operations of this nature, the method now pointed out should be followed; that is, *each quotient should be taken the nearest possible to the truth, whether it be too great or too small;* as by this means the numbers are kept smaller than they would otherwise be, and the work is simplified and shortened.

It may be farther remarked, that, in any such case, we are at liberty to change v into $-v$ wherever it occurs, if the change will serve any purpose: and the same may be done, in similar circumstances, in the next Chapter in the Diophantine Analysis.

ceding it, by adding 21, the coefficient of y; while the values of y, after the first are obtained by continual subtractions of 100, the coefficient of x. That this is universally true will appear in the following manner. Let $ax+by=c$ be the equation: then, by subtracting anb from the first member, and adding its equal bna to the remainder, we get

$$ax-anb+by+bna=c,$$

or, as it may be written,

$$a(x-nb)+b(y+na)=c.$$

From this it appears, that if from a value of x we take any number of times the coefficient of y, and at the same time add the same number of times the coefficient of x to the corresponding value of y; and if the quantities so obtained be substituted for x and y; the proposed equation will be satisfied: and therefore, if n be taken successively equal to 0, 1, 2, 3, &c., the values of x will form a decreasing arithmetical progression obtained by continual subtractions of b, and those of y an increasing one, got by continual additions of a.

229. If in the equation $ax+by=c$, we change b into $-b$, it becomes $ax-by=c$; and the expression found in the last § is changed into $a(x+nb)-b(y+na)=c$. From this it appears, that when the terms ax and by have opposite signs, and when any values of x and y have been found, other values of x will be obtained by continual additions of the coefficient of y; and of y by like additions of the coefficient of x. If n be taken negative, the additions will be converted into subtractions; so that the values of both x and y must go on all increasing or all diminishing, and not, as in the last §, the values of one of them increasing and those of the other diminishing: and hence it is plain, that if the equation $ax-by=c$ be such as to give any whole positive values for x and y, it will give an infinite number of such values.

230. The following question will serve as an example in reference to indeterminate equations of the kind just mentioned.

If one person have bank-notes worth £2 each, and another have notes worth 25 shillings each, how many must each give to the other, so that the former may pay to the latter a debt of 20 guineas?

Here, since £2 = 40*s*., and 20 guineas = £21 = 420*s*., if we put x for the number of notes given by the first, and y for the number repaid by the second, we shall have $40x-25y=420$;

or, by dividing by 5, $8x-5y=84$; and the work will stand as in in the margin. This operation so exactly resembles those exhibited and illustrated in § 227., and in the last note to that §, that it does not require explanation. In the values of x and y, as they come out in (9.) and (10.), v, in the algebraic sense, may be taken equal to any whole number whatever, whether positive or negative. In the plain meaning of the question, however, x and y must be positive; and it is evident, that this can be the case only when v is positive, and greater than 2. By taking it, therefore, successively equal to 3, 4, 5, &c., we get the following systems of answers, which, unlike those in § 227., are infinite in number:

v	3,	4,	5,	6,	7,	8,	&c.
x	13,	18,	23,	28,	33,	38,	&c.
y	4,	12,	20,	28,	36,	44,	&c.

$8x-5y=84$ .... (1.)

$y=\frac{8x-84}{5}$ .... (2.)

$y=2x-17-\frac{2x-1}{5}$ .. (3.)

$\frac{2x-1}{5}=v_1$ .... (4.)

$x=\frac{5v_1+1}{2}$ .... (5.)

$x=2v_1+\frac{v_1+1}{2}$ ..... (6.)

$\frac{v_1+1}{2}=v$ .... (7.)

$v_1=2v-1$ .... (8.)

$x=5v-2$.......... (9.)

$y=8v-20$ .... (10.)

231. As an example of another kind, let it be required to find a number, such that if it be divided successively by 5, 8, and 11, the respective remainders shall be 3, 6, and 9. To solve this, let x be the required number: then, if we assume $x=5y+3$, we shall satisfy the first condition, if y be a whole number; since, by dividing $5y+3$ by 5, we get y for quotient and 3 for remainder. In like manner, the assumptions, $x=8z+6$, and $x=11t+9$, will satisfy the remaining conditions, if z and t be whole numbers. To render these assumptions compatible with each other, let the first and second values of x be made equal: then, by transposition, $5y-8z=3$. Hence, by a process in every respect similar to those employed in the two preceding exercises, we get $z=5v-1$; and thence $x(=8z+6)=40v-2$; a value which satisfies the first and second conditions. Thus, if $v=1$, we have $x=38$, which gives 3 and 6 as remainders, when divided by 5 and 8. Now, by taking this expression equal to the third of the foregoing, and by transposition, we get $40v-11t=11$; whence, by the usual process for rendering v and t whole numbers,

we get $t=40v_2-1$; and therefore $x(=11t+9)=440v_2-2$; which, by taking v_2 successively equal to 1, 2, 3, &c., will be found to succeed. Thus, if $v_2=1$, we have $x=438$; and if this be divided successively by 5, 8, and 11, the remainders will be 3, 6, and 9. This is evidently the *least* positive number that will answer the conditions of the question; and it is plain that other answers will be found by successive additions of 440, the least common multiple of 5, 8, and 11.

232. By considering the solution given in § 227., it will be seen, that in equation (3.) the coefficient 100 was divided by the coefficient 21; that in (6.) 21 was divided by the remainder 16; that in (9.) 16 was divided by the remainder 5; and so on: and, as this process (the same as that for finding the greatest common measure of 21 and 100) is that which is employed in determining the fractions converging to the value of a continued fraction, we may naturally consider whether such indeterminate problems as those that have been before us may not admit of solution by means of the principles that have been established regarding continued fractions: and we shall find that they can be thus solved. To show how this may be effected, let us take the geneequation, $ax+by=\pm c$. Now, if we were to find by § 216. the fractions * converging to the ratio of a to b, and if we should denote the terms of the last but one by p and q, the last two would be $\frac{p}{q}$ and $\frac{a}{b}$: and if the former were taken from the latter, the numerator of the remainder would (§ 217.) be 1 or -1, according as $\frac{p}{q}$ should occupy an odd or an even place; that is $aq-bp=\pm 1$. Hence, by multiplying by $\pm c$, we get $a\times\pm qc+b\times\mp pc=c$; an equation which is the same as the original one, if x be taken equal to $\pm qc$, and y to $\mp pc$. These, therefore, are values of x and y; since, being substituted for them, they satisfy the assumed equation. Now, according to what was proved in § 228., the equation will still be satisfied, if we write $\pm qc+bv$ instead of $\pm qc$, and $\mp pc-av$ instead of $\mp pc$: and hence we have, as general values, $x=\pm qc+bv$, and $y=\mp pc-av$, where v is any whole number, positive or negative. Had the equation been $ax-by=c$, we should have got $x=\pm qc+bv$, and $y=\pm pc$

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* It should be recollected, that if a be less than b, the first quotient is to be taken $=0$.

$+av$. In both cases, the upper signs are to be used when $aq-pb=1$, and the lower when $aq-pb=-1$.

As an example, let us resume the equation $8x-5y=84$. Then, by dividing 8 by 5, 5 by the remainder, &c., we find the quotients 1, 1, 1, 2; and, by the process in the margin, we get $p=3$, and $q=2$. Then, by using the last pair of values given above for x and y, we get $x=2\times 84+5v=168+5v$, and $y=3\times 84+8v$ $=252+8v$. By changing v into $v-34$, which is plainly allowable, since v may be any whole number whatever, we get the simpler expressions, $x=5v-2$, and $y=8v-20$, the same as in § 230. It may be remarked that 34 is the nearest integral quotient obtained by dividing 168 by 5. By dividing in the value of y, 252 by 8, we get the quotients 31 and 32, which are equally near the truth.* If, therefore, we change v into $v-32$, we get $x=5v+8$, and $y=8v-4$; which expressions are also very simple.

$$1,\ 1,\ 1,\ 2;$$
$$\tfrac{1}{1},\ \tfrac{2}{1},\ \tfrac{3}{2},\ \tfrac{8}{5}.$$

233. If two equations containing three unknown quantities be given, one of the quantities may be eliminated, and the resulting equation treated in the manner that has been explained: or, generally, if there be m equations containing $m+1$ unknown quantities, all these quantities may be eliminated except two, and then the work will proceed in the manner already explained.

If, for example, the two equations, $3x+5y+2z=40$, and $4x+4y+z=33$, be proposed, by taking the former from double the latter we get $5x+3y=26$. Then, by the usual process, we find $x=1+3v$, and $y=7-5v$: and by substituting these in either of the given equations (the second rather), we get $z=1+8v$. By taking, then, $v=0$, and $v=1$, we get $x=1$, $y=7$, $z=1$; and $x=4$, $y=2$, $z=9$; which are the only positive answers to the question.

234. If there be only one equation given containing three unknown quantities, we may assign to one of them any value we please, and then, in the usual way, find the corresponding values of the others: after that we may assign another value to the first, and proceed as before; and so on: and it is easy to see that the

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* This simplification will be obtained universally, by dividing qc by b, attaching the quotient to v with the sign opposite to that of bv, and changing v into the result: and a like simplification would be obtained from the expressions in the value of y. Other modifications will readily suggest themselves in particular cases; and the general rule to be followed is to employ as small numbers as possible.

same plan may be followed in case of a still greater number of unknown quantities.

To exemplify this, let us take the equation, $2x+3y+5z=41$. Now, if $z=1$, this gives $2x+3y=36$; whence the values of x and y may be found. If, again, $z=2$, we have $2x+3y=31$; whence x and y may be determined: and thus we may proceed as far as we please. The following twenty-one answers, however, are the only ones that are all positive.

$$z=1: \begin{cases} x=15,\ 12,\ 9,\ 6,\ 3; \\ y=\ 2,\ 4,\ 6,\ 8,\ 10. \end{cases}$$

$$z=2: \begin{cases} x=14,\ 11,\ 8,\ 5,\ 2; \\ y=\ 1,\ 3,\ 5,\ 7,\ 9. \end{cases}$$

$$z=3: \begin{cases} x=10,\ 7,\ 4,\ 1; \\ y=\ 2,\ 4,\ 6,\ 8. \end{cases}$$

$$z=4: \begin{cases} x=9,\ 6,\ 3; \\ y=1,\ 3,\ 5. \end{cases}$$

$$z=5: \begin{cases} x=5,\ 2; \\ y=2,\ 4. \end{cases}$$

$$z=6: \begin{cases} x=4,\ 1; \\ y=1,\ 3. \end{cases}$$

*Exercises.** Find the integral values of x and y in the following equations.

1. $2x+3y=25$.

$$Ans. \begin{cases} x=\ldots\ -4,\ -1,\ 2,\ 5,\ 8,\ 11,\ 14,\ 17,\ldots \\ y=\ldots\ 11,\ 9,\ 7,\ 5,\ 3,\ 1,\ -1,\ -3,\ldots \end{cases}$$

2. $5x+7y=52$.

$$Ans. \begin{cases} x=\ldots\ -12,\ -5,\ 2,\ 9,\ 16,\ 23,\ldots \\ y=\ldots\ 16,\ 11,\ 6,\ 1,\ -4,\ -9,\ldots \end{cases}$$

3. $4x+13y=229$.

$$Ans. \begin{cases} x=\ldots\ -11,\ 2,\ 15,\ 28,\ 41,\ 54,\ 67,\ldots \\ y=\ldots\ 21,\ 17,\ 13,\ 9,\ 5,\ 1,\ -3,\ldots \end{cases}$$

4. $3x+5y=7$. $$Ans. \begin{cases} x=\ldots\ -6,\ -1,\ 4,\ 9,\ldots \\ y=\ldots\ 5,\ 2,\ -1,\ -4,\ldots \end{cases}$$

5. $7x-9y=5$. $$Ans. \begin{cases} x=\ldots\ -16,\ -7,\ 2,\ 11,\ 20,\ldots \\ y=\ldots\ -13,\ -6,\ 1,\ 8,\ 15,\ldots \end{cases}$$

6. $8x-7y=1$. $$Ans. \begin{cases} x=\ldots\ -13,\ -6,\ 1,\ 8,\ 15,\ldots \\ y=\ldots\ -15,\ -7,\ 1,\ 9,\ 17,\ldots \end{cases}$$

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* The student will perceive, that the first of the following exercises has four solutions in positive numbers, the second two, the third five, the fifth and sixth an infinite number, and the fourth none. The seventh, also, has five such solutions, and the eighth seven. We may obtain as many solutions as we please, containing negative values, by continuing the arithmetical progressions backward or forward, or both.

7. $\begin{cases} x+3y+5z=44, \\ 3x+5y+7z=68. \end{cases}$

$$Ans. \begin{cases} x= \dots\dots -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ \ 5, \dots\dots \\ y= \dots\dots\ 10,\ 8,\ 6,\ 4,\ 2,\ 0,\ -2, \dots\dots \\ z= \dots\dots\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ \ 9, \dots\dots \end{cases}$$

$2x+3y+4z=21.$

$$Ans.\ z=1;\ \begin{cases} x=7,\ 4,\ 1; \\ y=1,\ 3,\ 5. \end{cases} \Bigg|\ z=2:\ \begin{cases} x=5,\ 2; \\ y=1,\ 3. \end{cases}$$

$$z=3:\ \begin{cases} x=3; \\ y=1. \end{cases} \Bigg|\ z=4:\ \begin{cases} x=1; \\ y=1. \end{cases}$$

Exer. 9. In how many different ways may £150 be paid in dollars of 4*s.* 6*d.* each and guineas?

Ans. In forty-eight different ways.*

10. A boy, having gathered a quantity of nuts, found that when he counted them by twos, threes, fours, fives, or sixes, there was always one remaining; but when he counted them by sevens, there was no remainder. How many had he?

Ans. 301, or $301+420v$.

11. Divide 200 into two parts, such that if one of them be divided by 6, and the other by 11, the respective remainders may be 5 and 4.† *Ans.* 119 and 81, or 53 and 147.

CHAPTER XIV.

DIOPHANTINE ANALYSIS.

235. THE object of another branch of the indeterminate analysis is to find such values of quantities as shall render functions of them exact powers, such as squares or cubes; or, what amounts

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* In solving this we get $x=3v+1$, and $y=662-14v$. Hence, the greatest value of y is 662, and the others will be found by continual subtractions of 14. Now, by dividing 662 by 14, we get 47, with the remainder 4; and hence it appears, that, in addition to 662, there may be 47 other positive answers.

† To solve this, we may take x and y to denote the quotients; then the parts will be $6x+5$ and $11y+4$; and by the question, $6x+5+11y+4=200$, whence $6x+11y=191$.

to the same, to find such values of a quantity, as shall render a radical depending on it rational. Thus, for instance, it might be required to find such values of x (2, 5, &c.) as would render $6-x$ an exact square, so that $\sqrt{(6-x)}$ could be exactly determined. This is generally termed the *Diophantine analysis*, because the mathematicians of modern times derived their knowledge of the subject from an able work by Diophantus of Alexandria in Egypt, who is believed to have lived about the middle of the fourth century of the Christian era. While this subject, in its full extent, presents great, and in many instances, insuperable difficulties, yet the most useful inquiries connected with it are conducted with much facility. The most important of these will form the subject of the present Chapter.

236. Let us first consider the method of assigning such values to x, that the expression, ax^2+bx+c, which, for brevity, we may denote by X, may be an exact square, or, which is the same, that the square root of ax^2+bx+c may be exactly taken. Now, the mode of solution will depend on the individual or relative values of a, b, and c, particularly those of a and c. Thus, we may have $a=0$, $c=0$, a a square number (a'^2), or c a square number (c'^2). Let us, therefore, consider these cases.

237. If $a=0$, X becomes $bx+c$; and if we put this equal to v^2, we shall get $x=\frac{v^2-c}{b}$, where v may be taken of any value.

Exam. 1. If $3x-2$ be proposed to be made a square, we shall have $x=\frac{1}{3}(v^2+2)$; and therefore, if $v=1, 2, 3$, &c., we shall find $x=1, 2, 3\frac{2}{3}$, &c.; which will change $3x-2$ into 1, 4, 9, &c., each of which has an exact square root, the same as the value assumed for v.

238. If $c=0$, we have $X=ax^2+bx$. Put this equal to v^2x^2: then, by dividing by x, we get $ax+b=v^2x$; whence, by transposition and division, we find $x=\frac{b}{v^2-a}$.

Exam. 2. Let it be required to find values of x, each of which will make $2x^2-3x$ a square. Here $a=2$ and $b=-3$; and therefore $x=\frac{-3}{v^2-2}=\frac{3}{2-v^2}$. In this v may be taken of any value except $\sqrt{2}$. If, however, we wish to have x positive, we must take v between $\sqrt{2}$ and $-\sqrt{2}$. Thus, if we take $v=1$, we get

$x = 3$, and $\sqrt{(2x^2-3x)}\ (=vx) = 3$; and if $v = \frac{1}{2}$, we find $x=\frac{12}{7}$, and $\sqrt{(2x^2-3x)}=\frac{6}{7}$.

239. If a be a square, putting its root $= a'$, we shall have $X=a'^2x^2+bx+c$. Let this be put $=(a'x+v)^2=a'^2x^2+2a'vx+v^2$. Then, by rejecting a'^2x^2, we get $bx+c=2a'vx+v^2$; whence $x=\frac{c-v^2}{2a'v-b}$.

Exam. 3. Let it be required to render $4x^2+3x-7$ an exact square. Here $a' = 2$, $b = 3$, and $c = -7$; and, consequently, $x=\frac{-7-v^2}{4v-3}=\frac{7+v^2}{3-4v}$. This will give x positive, when v is less than $\frac{3}{4}$. Thus, if $v = \frac{1}{2}$, we get $x=7\frac{1}{4}$, and $\sqrt{X}=15$: also, if $v = -1$, we have $x=\frac{8}{7}$, and $\sqrt{X}=\frac{9}{7}$; and if $v = -2$, we get $x=1$, and $\sqrt{X}=0$.

240. When c is a square, denote it by c'^2, and we have $X=ax^2+bx+c'^2$. Assume this $= (vx+c')^2=v^2x^2+2c'vx+c'^2$: then, by rejecting c'^2, there is obtained $ax^2+bx=v^2x^2+2c'vx$; from which, by dividing by x, and by transposing, &c., we get

$$x=\frac{b-2c'v}{v^2-a}.$$

Exam. 4. To find values of x that will make $\sqrt{(-2x^2+5x+9)}$ rational, we have $a = -2$, $b = 5$, and $c' = 3$; and, therefore $x=\frac{5-6v}{v^2+2}$. Then, if $v = 1$, we get $x=-\frac{1}{3}$, and $\sqrt{X}=\frac{8}{3}$; and if $v = -1$, we find $x=\frac{11}{3}$, and $\sqrt{X}=\frac{2}{3}$.

241. When the values of a and c are neither nothing nor squares, the foregoing methods of solution fail. Even then, however, the solution can be effected in particular cases. One of these is that in which X is the product of rational factors. Now, by putting $ax^2+bx+c=0$, resolving (§ 151 or 152.) the equation so obtained, and (§ 161.) attaching the roots with their signs changed to x, we find that X is $=$

$$a\left\{x+\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a}\right\}.\left\{x+\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}\right\}:$$

and this expression will evidently be rational throughout when

b^2-4ac is a square number. In this case, by denoting the quantities annexed to x by $-x'$ and $-x''$, we get $ax^2+bx+c=a(x-x')(x-x'')$: and by putting this equal to $v^2(x-x'')^2$, and dividing by $x-x''$, we obtain $a(x-x')=v^2(x-x'')$; whence $x=\frac{v^2x''-ax'}{v^2-a}$.

Exam. 5. If it be required to find values of x, each of which will make $3x^2-8x+5$ a square, though none of the former principles is applicable, yet, since $8^2-4\times3\times5$, is equal to 4, a square, the solution can be thus effected on the principle just established. By resolving the equation, $3x^2-8x+5=0$, we get $x'=\frac{5}{3}$ and $x''=1$; and therefore we have $x=\frac{v^2-5}{v^2-3}$. Hence, by taking $v=1$, we get $x=2$, and $\sqrt{X}=1$; and, by taking $v=5$, we find $x=\frac{10}{11}$, and $\sqrt{X}=\frac{5}{11}$.

242. If none of the foregoing methods be applicable, still the solution can be effected, if X be equal to the square of a simple rational factor increased or diminished by the product of two others; that is, if $X=(a_1x+b_1)^2\pm(a_2x+b_2)(a_3x+b_3)$. In this case we are to assume the given quantity equal to $\{(a_1x+b_1)+v(a_2x+b_2)\}^2$: then, by actually squaring, rejecting $(a_1x+b_1)^2$, dividing by a_2x+b_2, &c., we should determine the value of x.

Exam. 6. Required a number such, that five times its square may exceed another square by 1. Here, if x be the number, we shall have $5x^2-1$ equal to a square. We have, therefore, $a=5$, $b=0$, $c=-1$, and consequently $b^2-4ac=20$. Now, this last not being a square, and neither a nor c being nothing or a square, we cannot employ any of the former principles. It is seen at once, however, that $5x^2-1$ may be put under the form $(2x)^2+(x-1)(x+1)$. Assuming this therefore equal to $\{2x+v(x-1)\}^2$, we readily find $x=\frac{v^2+1}{v^2+4v-1}$. Hence, if $v=\frac{1}{2}$, we get $x=1$, and $X=4=2^2$: and, by taking $v=1$, we find $x=\frac{1}{2}$, and $X=\frac{1}{4}=(\frac{1}{2})^2$.

243. When all the foregoing methods fail, we may often find by trial a value x' of x that will render X a square. Then, if $y+x'$ be substituted in X for x, an expression will be obtained from which as many values as we please may be found for y, and

thence an equal number of corresponding values for x. To prove this, substitute, first x' and then $y+x'$ for x in X, and there will be obtained $ax'^2+bx'+c$, and $ay^2+(2ax'+b)y+ax'^2+bx'+c$. Now, the first of these being a square, if we denote it by c'^2, and substitute this for it in the second, the latter will become $ay^2+(2ax'+b)y+c'^2$; which, since the last term is a square, may be made a square by means of § 240.

Exam. 7. Let it be required to find x such that $6x^2-10x-3$ may be a square. Here, by trying some of the smaller numbers, 1, -1, 2, &c., we find, that when $x=2$, the proposed quantity becomes $24-20-3=1$, a square. Let, therefore, $x=y+2$, and X will become $6(y+2)^2-10(y+2)-3$, or, by contraction, $6y^2+14y+1$. Hence, by putting this equal to $(vy+1)^2$, we find $y=\dfrac{2v-14}{6-v^2}$; and thence $x=y+2=\dfrac{2v^2-2v+2}{v^2-6}$. From this we may find as many values of x as we please. Thus, if we take $v=0$, 1, 2, &c., we shall find the corresponding values of x to be as in the margin.*

$v=$	0,	1,	2,	3,	4,	-3,	&c.
$x=$	$-\frac{1}{3}$,	$-\frac{2}{5}$,	-3,	$\frac{14}{3}$,	2·6,	$\frac{26}{3}$,	&c.

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* When the principles pointed out above fail in giving a solution, it is very probable that the proposed quantity cannot be made a square. At the same time, such an inference ought not to be hastily drawn; and, to complete the theory, it would be necessary to investigate the cases in which X *cannot* be made a square. An inquiry of this kind, however, would be unsuitable to the nature and extent of the present work; but those who wish to study this branch of the subject may consult Euler's *Algebra*, vol. ii. chap. v.; Barlow's *Theory of Numbers*, chap. iv., and other works. The following investigations, in which it is proved that no rational value whatever, whole or fractional, can be assigned to x so as to make $3x^2+2$ a square, afford an instance of the kind referred to.

Since, in dividing any number by 3, the remainder must be either 0, or 1, or 2, it follows that every number must be of one of the three forms, $3n$, $3n+1$, $3n+2$; and therefore, by squaring these, we find that the square of every number is of one of the three forms,

$$9n^2, \qquad 9n^2+6n+1, \qquad \text{and } 9n^2+12n+4.$$

Now, it will be seen that, if the second and third of these be divided by 3, there will be 1 remaining; and, also, that the first is divisible not only by 3 but by 9. Since, also, none of the expressions gives 2 as remainder, it follows that no square number can be of the form, $3n+2$, but must be of one of the two forms, $9n$ and $3n+1$.

It may be shown in the next place, that p and q being prime to one another, there can be no square number of the form, $3p^2+2q^2$. To prove

244. We may now proceed to consider how expressions of the two forms, $\sqrt{X}=\sqrt{(ax^3+bx^2+cx+d)}$ and $\sqrt{X}=\sqrt{(ax^4+bx^3+cx^2+dx+e)}$, can be rendered rational; problems which present much more difficulty than the foregoing, and which admit of a much more limited number of solutions. In the solution of these, the general principle is, to assume the given expression

this, since, as we have just seen, squares can have only the forms, $9n$ and $3n+1$, let us suppose p^2 to have either of the two values, $9n$ and $3n+1$, and q^2 either of the two, $9m$ and $3m+1$. Now, the condition, that p and q are prime to each other, precludes our taking, simultaneously, $p^2=9n$, and $q^2=9m$, as these have the common factor 9. The only combinations remaining, therefore, are $p^2=9n$ and $q^2=3m+1$; $p^2=3n+1$ and $q^2=3m+1$; and $p^2=3n+1$ and $q^2=9m$. The first of these gives $3p^2+2q^2=27n+6m+2$; the second $9n+3+6m+2$; and the third $9n+3+18m$. Now, if the first and second of these results be divided by 3, there is in each case the remainder 2; while the third is divisible by 3, and not by 9; and therefore, by what we have seen, $3p^2+2q^2$ cannot be a square.

It is plain, from § 98., that if one square be divided by another, the quotient is a square, and that if a quantity which is not a square be divided by one which is, the quotient is not a square; since, in each instance, the root of the fraction so obtained is equal to the root of the numerator divided by that of the denominator. Hence, if we divide $3p^2+2q^2$ by q^2, and denote the fraction whose terms are p^2 and q^2 by x^2, we shall come to the conclusion, that, universally, $3x^2+2$ cannot be a square, whether x, as formerly, be a whole number, or, as now, a fraction. It is scarcely necessary to say that the quantities, n, m, p, &c., employed above are whole numbers, with the exception of x, when used at the conclusion for the fraction whose terms are p and q.

It might be shown in a similar manner, that none of the expressions, $3p^2+5q^2$, $3p^2+8q^2$, ..., $3p^2+(3n+2)q^2$, can be a square.

Similar reasonings and investigations would be admissible, were 4, 5, or any other whole number taken as the divisor of other numbers. Thus, by taking 4, we find that all numbers are of one or other of the four forms $4n$, $4n+1$, $4n+2$, and $4n+3$; and, by squaring these, we should find, that all the even square numbers are of the form, $16n$ or $16n+4$; and that the odd ones are of the form, $8n+1$: and hence it will follow, that no number of the form, $8n+3$, $8n+5$, $8n+7$, $16n+2$, $16n+6$, $16n+8$, $16n+10$, $16n+12$, or $16n+14$, can be a square number.

On similar principles, since every number is of one of the forms, $5n$, $5n+1$, $5n+2$, $5n+3$, and $5n+4$, it may be shown, by squaring these, that every square number is of the form $25n$, $5n+1$, or $5n+4$, or, what is equivalent, $25n$, $5n+1$, and $5n-1$; and that, in consequence, no number of the form $5n+2$ or $5n+3$ can be a square; and thus we know at once that 502, 503, 1002, 1003, &c., are not squares. It follows, also, from this property, that all square numbers end in 5, 0, 1 $(=0+1)$, 9 $(=10-1)$, 4 $(=5-1)$, or 6 $(=5+1)$. Thus, the squares of the nine digits 1, 2, 3, &c., are 1, 4, 9, 16, 25, 36, 49, 64, and 81.

equal to such a quantity, that, after squaring, all the terms may disappear, or may be made to disappear, except those containing two consecutive powers of x; as the value of this quantity will then be obtained in a rational form. The terms to be destroyed may be at the beginning of the given expression, at its end, or at both, according to its nature. The following examples will illustrate a number of the most useful modes of effecting the solution.

Exam. 8. Let it be required to render the expression $\sqrt{(2x^3-5x^2+12x+4)}$ rational. Here, since 4 is a square, let us assume $\sqrt{X}$ equal to $vx+2$. Then, by squaring, and rejecting 4, we get $2x^3-5x^2+12x=v^2x^2+4vx$. Now, if in this we take $4v=12$, or $v=3$, the terms $12x$ and $4vx$, will destroy one another, and v^2 becoming 9, we have $2x^3-5x^2=9x^2$; whence $x=7$ which is found to answer, as, by its substitution for x, the given quantity $\sqrt{X}$ becomes 23.

This may also be solved by assuming $\sqrt{X}$ equal to the trinomial, $v'x^2+vx+2$. On this assumption, by squaring both members, we get

$$2x^3-5x^2+12x+4=v'^2x^4+2vv'x^3+(v^2+4v')x^2+4vx+4.$$

Hence, by rejecting 4, taking $4v=12$, and $v^2+4v'=-5$, we find $v=3$, and $v'=-\frac{7}{2}$; and, by rejecting $-5x^2$ and $12x$, and their equals, $(v^2+4v')x^2$ and $4vx$, we have

$$2x^3=v'^2x^4+2vv'x^3=\tfrac{49}{4}x^4-21x^3.$$

From this, by dividing by x^2, &c., we get $x=\frac{92}{49}$, which also answers.

Exam. 9. What value of x will make $4x^4+12x^3-3x^2-2x+1$ a square?

By assuming this equal to $(2x^2+vx+v')^2$, actually squaring, and rejecting $4x^4$, we get

$$12x^3-3x^2-2x+1=4vx^3+(v^2+4v')x^2+2vv'x+v'^2.$$

Then, by equating the coefficients 12 and $4v$, and also -3 and v^2+4v', we get $v=3$, and $v'=-3$; and the equation is reduced to $-2x+1=2vv'x+v'^2=-18x+9$; whence $x=\frac{1}{2}$.

We may also solve this by assuming X equal to $(vx^2+v'x+1)^2$; as, by actually squaring, and rejecting 1, we obtain the equation,

$$4x^4+12x^3-3x^2-2x=v^2x^4+2vv'x^3+(v'^2+2v)x^2+2v'x:$$

and by equating the coefficients -2 and $2v'$, and also -3 and v'^2+2v, we find $v'=-1$, and $v=-2$. Hence the equation becomes simply

$$4x^4+12x^3=v^2x^4+2vv'x^3=4x^4+4x^3;$$

and therefore $x=0$, which answers. The result might have been obtained at once by inspection; as when the last term, as in the present case, is a square, the quantity will evidently be rendered a square by taking $x=0$.

As a third mode of solution, we may assume $\sqrt{X}=2x^2+vx+1$, the first and last terms being the square roots of the first and last terms of X. Then, by actually squaring, and by rejecting the first and last terms, we get

$$12x^3-3x^2-2x=4vx^3+(v^2+4)x^2+2vx.$$

Now, this may be resolved in two ways, either by destroying the first term or the last, in each member. The first is effected by taking $4v=12$, and x is then found to be $-\frac{1}{2}$. To take away the last terms we must have $2v=-2$; and on this supposition we find x to be $\frac{1}{2}$, the same that was found by the first method.

245. It is plain, that none of the foregoing methods can be employed, unless, in an expression of the third degree, the last term be a square; or, in one of the fourth, at least one of the two extreme terms be such. In that case, if we can find by trial a value x' for x, which answers, we may substitute, as in § 236., $y+x'$ for x, and we shall thus obtain an expression having its last term a square, and which may therefore be made a square by one of the methods employed already.

Exam. 10. Let it be required to render $3x^4-2$ a square.

Here we see at once, that this will become a square if $x=1$. By substituting, therefore, $y+1$ for x in $3x^4-2$, we get $3y^4+12y^3+18y^2+12y+1$, the last term of which is a square. Then, by assuming this equal to $(vy^2+v'y+1)^2$, we find that the last three terms in each member disappear by taking $v'=6$, and $v=-9$; and we readily get $y=\frac{20}{13}$, and thence $x=\frac{33}{13}$, which is found to answer, making $3x^4-2$ a square. Were we now to assume $x=y+\frac{33}{13}$, and to follow out a similar process, we should obtain another value of x; and thus we might proceed as far as we please, finding value after value for x. These, however, would soon come to be expressed in numbers of extremely great and inconvenient magnitude.*

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* We saw in the note to § 243., that innumerable expressions of the

246. Quantities of the form, $X=ax^3+bx^2+cx+d$, can be made cubes, or, which is the same, their cube roots can be made rational, on principles exactly similar to those that have been employed in rendering quantities squares. Thus, if a be a cube, we may destroy the first and second terms: if d be a cube, the third and fourth terms can be destroyed: if a and d be both cubes, we can destroy the first and last terms: and, when neither a nor d is a cube, if we can find a value x', which, being substituted for x, will render X a cube, we may substitute $y+x'$ for x, and we shall get an expression which can be rendered a cube by the second of the principles just mentioned. The processes thus indicated will be understood from the following examples.

Exam. 11. Let it be required to find a value of x which will render $8x^3+12x^2+27x-27$ a cube. By assuming the root of this equal to $2x+v$, cubing, and rejecting the first terms, we get $12x^2+27x-27=12vx^2+6v^2x+v^3$. We then destroy the first terms of this by taking $12v=12$, and therefore $v=1$: and the equation becomes simply $27x-27=6x+1$; whence $x=\frac{4}{3}$, which answers, giving $X=\frac{1331}{27}$, the cube root of which is $\frac{11}{3}$.

For a second solution, we may assume the root $=vx-3$. Then, by putting the cube of this equal to X, and taking $v=1$, the equation is reduced to $8x^3+12x^2=x^3-9x^2$; whence $x=-3$; which also answers, giving $X=-216=(-6)^3$.

For a third solution, we may assume $X=(2x-3)^3$, $2x$ and -3 being the cube roots of the first and last terms. Then, by actually cubing, and rejecting the equal terms, we get $12x^2+27x$

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second degree cannot be made squares; and, as may be readily supposed, there are numberless expressions of the same kind belonging to the third and fourth degrees. When expressions of these degrees cannot be made squares by some of the methods that have been pointed out, there is at least reason to presume that they cannot be made such. It can be demonstrated also, though the limits of the present work preclude the attempt, that none of the following expressions, besides innumerable others, can be made rational, unless either x or y be nothing; $\sqrt{(x^4+4y^4)}$, $\sqrt{(x^4-4y^4)}$, $\sqrt{(4x^4-y^4)}$, $\sqrt{(2x^4+2y^4)}$, $\sqrt{(2x^4-2y^4)}$, and $\sqrt{(x^4+2y^4)}$ See Euler's *Algebra*, Part II. chap. 13.

It may be farther remarked, that, by no means at present known, can expressions of a higher degree than the fourth be made squares. Thus, if there were an expression of the fifth degree, and if we assumed as its root either $vx^2+v'x+v''$, or $vx^3+v'x^2+v''x+v'''$, it would be found that we could not destroy so many terms as to leave an equation containing only two consecutive terms; and therefore the values of x found from the resulting equation would be generally irrational.

$=-36x^2+54x$; whence $x=\frac{9}{16}$, which is also a correct answer, giving $X=-\frac{27000}{16^3}=(-\frac{15}{8})^3$.

Exam. 12. Let it be required to find a value of x which will make $2x^3+3$ a cube. Here, we readily see that x may be -1, as X then becomes $-2+3$ or 1, the cube of 1. We take, therefore, $x=y-1$, and by this means X becomes $2y^3-6y^2+6y+1$. Then, by assuming this equal to $(vy+1)^3$, cubing, and taking $v=2$, so as to make the last two terms of each member disappear, we get $y=-3$, and consequently $x=-4$, which answers, giving $X=-125=(-5)^3$. Were we to substitute $y-4$ for x in X, and proceed as before, we should find another value of x; and from it we might derive another, and so on.

247. We may now briefly consider what have been called *double, triple, or higher equalities*; that is, problems in which two, three, or more functions of a quantity are to be made squares or cubes at the same time. Thus, if it be required to find a value of x, which shall render both the expressions, a_1x+b_1 and a_2x+b_2, squares, the problem presents a double equality. To solve this, we may put the first expression equal to x'^2, and find the value of x from the equation so obtained. Then, if this value be substituted for x in the second expression, it will only remain to find such a value of x' as shall make the result a square, which will be done by some of the methods already explained. This will be exemplified in the solution of the following question.

Exam. 13. Required a number, such that if 1 be taken from itself, and the same from its double, each remainder shall be a square. Here, putting x for the required number, we have to make $x-1$ and $2x-1$ squares. Assuming the first equal to x'^2, we get $x=x'^2+1$; and by substituting this for x in the second, we obtain $2x'^2+1$, which is to be made a square. For the root of this we assume (§ 240.) $vx'+1$, and we readily find

$$x'=\frac{2v}{2-v^2}, \text{ and thence } x=x'^2+1=\frac{4+v^4}{(2-v^2)^2}, \text{ or } x=\frac{v^4+4}{(v^2-2)^2};$$

the latter two expressions being identical, since the squares of $2-v^2$ and v^2-2 are the same. By taking $v=1$, we get $x=5$, which answers; since $5-1=4=2^2$, and $2\times5-1=9=3^2$. In like manner, by taking 3 and 4, we get $x=\frac{85}{49}$, and $x=\frac{65}{49}$; and, by making other assumptions for v, we may get as many values for x as we please.

248. In case of the triple equality, a_1x+b_1, a_2x+b_2, and a_3x+b_3, we may find, as in the last §, the value of x which will render the first and second expressions squares. Then, by substituting this in the third, we get an expression of the fourth degree, which will be made a square by means of § 244., if it admit of being made such.

Exam. 14. Required a number, such that if 1 be taken from it, 2 be added to it, and 1 be added to its quadruple, the results shall all be squares.

In this question the three quantities to be made squares are $x-1$, $x+2$, and $4x+1$. Then, by taking $x-1=x'^2$, and proceeding as in the last example, we find $x=\frac{v^4-2v^2+9}{4v^2}$ to be the value which will render the first and second expressions squares. By substituting this in the third, $4x+1$; rejecting $4v^2$, the denominator of the result, as it is a square, and dividing by the square 4, we get v^4-v^2+9, which is to be a square. Now, we readily see that this will be a square, if $v=1$: and, therefore, (§ 236.) we substitute $y+1$ for v. By this means we get $y^4+4y^3+5y^2+2y+9$: and, if we assume $y^2+v'y+3$ as the root of this, and take $v'=2$, we find $y=-2$; and consequently $v=-1$, and $x=2$, which answers. If, instead of taking $v'=2$, we had taken it $=\frac{1}{3}$, we should have got $y=\frac{1}{3}$, $v=\frac{4}{3}$, and $x=\frac{697}{576}$, another answer: and by means of these answers we might, as in § 238., find others.

249. In case, again, of the double equality, $a_1x^2+b_1x$, and $a_2x^2+b_2x$, we assume the first equal to v^2x^2. Then, by finding from this the value of x, and substituting it in the second, we find a quantity which may be made a square by some of the methods already explained. As another method, we may change x into y^{-1} in both: and then, by multiplying the results by the square y^2, we get a_1+b_1y and a_2+b_2y; which may be rendered squares in the way pointed out in § 247.

250. If it be required to make squares of $a_1x^2+b_1x+c_1$ and $a_2x^2+b_2x+c_2$, let such a value of x be found (§ 236, &c.) as shall make one of them a square. Then, by substituting that value of x in the other, and modifying the result, an expression will be found, which will be of the fourth degree, and which may be made a square, if possible, by means of § 244.*

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* A considerable outline of what is known regarding the Diophantine

Miscellaneous Examples.

1. To divide a given square number a^2 into two squares.*

To solve this, let x^2 be one of the parts: then a^2-x^2 will be the other. To make this a square, assume it (§ 240.) $= v^2(a-x)^2$ †: then, dividing by $a-x$, we get $a+x=v^2(a-x)$; and therefore by resolving for x, we find

$$x=\frac{v^2-1}{v^2+1}.a: \text{ whence } x^2=\frac{(v^2-1)^2}{(v^2+1)^2}.a^2; \text{ and } a^2-x^2=\frac{4v^2}{(v^2+1)^2}.a^2:$$

the subtraction being performed by multiplying a^2 by $(v^2+1)^2$, and taking the product as numerator, and $(v^2+1)^2$ as denominator, and then actually squaring and subtracting. As particular instances, suppose a^2 to be 100: then if $v=2$, we find the parts to be 36 and 64: if $v=4$, they are $\frac{22500}{289}$ and $\frac{6400}{289}$; and so on.

Hence, if we divide a^2 and its two parts by the square a^2, and multiply the quotients by the square $(v^2+1)^2$, we find that $(v^2+1)^2=(v^2-1)^2+4v^2$. If, in this, we change v into pq^{-1}, and multiply the results by q^4, we get $(p^2+q^2)^2=(p^2-q^2)^2+4p^2q^2$. Hence, the square of p^2+q^2 being equal to the sum of the squares of p^2-q^2 and $2pq$, it follows (*Euc.* I. 48.), that if p^2+q^2 be the hypotenuse of a right-angled plane triangle, p^2-q^2 and $2pq$ will be its legs; and therefore we have the fol-

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analysis having now been given, the following examples and exercises are subjoined for the purpose of illustrating it. In solving such problems, the student will find that much, with regard to simplicity and elegance, depends on address in determining on the plan of the solutions, and in particular in making such assumptions as may shorten and simplify the work. The whole subject, though it does not contribute much to the advancement of the great and important branches of science, is still of no small curiosity and interest; and accordingly it has attracted the attention and exercised the minds of several of the most distinguished mathematicians of modern times, particularly Euler, Lagrange, Legendre, and Gauss; whose works, as well as those of several other writers, and various articles in mathematical periodicals, may be consulted by those who wish to become extensively acquainted with the subject.

* The solution of this question enables us to divide a given square into any assigned number of squares; as we have only to divide it first into two squares, and then subdivide one or both of these into others.

To solve the problem in which it is required to find three or more square numbers, whose sum shall be a square, not necessarily a given one, assume at pleasure a^2, b^2, c^2, &c., to represent them all but one; then, taking x^2 to represent that one, we have merely to render $x^2+a^2+b^2+$ &c., a square; which will be done by assuming it equal to $(x+v)^2$.

† We might assume the root $= a+vx$, according to § 239.

lowing rule, which is often useful: *To find the sides of a right-angled triangle in whole numbers, take two unequal whole numbers: then the sum of their squares, the difference of their squares, and twice their product will be the three sides.* Thus, by assuming 1 and 2, we find the sides to be 5, 3, and 4; while, by assuming 2 and 5, we get 29, 21, and 20.

2. Required two square numbers, whose difference shall be a given number a. Let the less number be x^2: then the greater will be x^2+a. To render this a square, assume it equal to $(x+v)^2$, and it will be easily found that $x=\frac{a-v^2}{2v}$. Thus, suppose $a=12$: then, if $v=2$, we get $x=2$, $x^2=4$, and $x^2+a=16$, so that 4 and 16 are the numbers. Also, if $v=4$, we have $x=-\frac{1}{2}$, $x^2=\frac{1}{4}$, and $x^2+a=12\frac{1}{4}=(3\frac{1}{2})^2$.

3. To divide a number which is the sum of two known squares, a^2 and b^2, into two other squares. To effect this, let x^2 be one of the parts: then $a^2+b^2-x^2$ is to be a square. Now, one satisfactory value of x is b (or a); and, therefore, by substituting (§ 236.) $y+b$ for x, we get a^2-y^2-2by; by assuming which equal to $(a-vy)^2$, we find

$$y=\frac{2av-2b}{v^2+1}, \text{ and } x=y+b=\frac{bv^2+2av-b}{v^2+1}=\frac{b(v^2-1)+2av}{v^2+1}.$$

As instances, let the given number be $185=4^2+13^2$, so that $a=4$ and $b=13$: then, by taking 2 as the value of v, we shall find $x=11$, and $x^2=121$; and also $185-121=64=8^2$. If $v=4$, we find the corresponding parts to be $(\frac{227}{17})^2$ and $(\frac{44}{17})^2$. It is plain that we might have taken $a=13$, and $b=4$; and we should thus have obtained other answers.

4. Let it be required to find three square numbers in arithmetical progression. Since $(x-y)^2$, x^2+y^2, and $(x+y)^2$, are plainly in arithmetical progression, having the common difference $2xy$, and since the first and third are already squares, it is only necessary to find x and y such that x^2+y^2 may be a square; and this will be done in the manner pointed out in the concluding part of the solution to Exam. 1. Thus, by employing the two numbers 2 and 1, we get x (the difference of their squares) $=3$, and y (twice their product) $=4$; and therefore $(x-y)^2$, x^2+y^2, $(x+y)^2$ are 1, 25, and 49, which answer. If, again, we take 3 and 2, we get $x=5$, and $y=12$; and consequently $(x-y)^2$, x^2+y^2, and $(x+y)^2$ are 49, 169, 289, which also answer.

5. To find any assigned number (n) of squares, whose sum shall be a square.

Here, by assuming as the required squares a_1^2, a_2^2, a_3^2, ..., a_{n-1}^2, and x^2, where a_1, a_2, &c., are numbers assumed at pleasure, it only remains to find such a value of x as shall make $a_1^2+a_2^2+\ldots.+x^2$ a square, which (§ 239.) will be effected by assuming it equal to $(x+v)^2$, and resolving the equation, so found, for x.

6. To find a number, such that if 1 be added to its square, the double of the result will be a square.

By taking x as the required number, we have to make $2x^2+2$ a square, which cannot be effected by any of the methods given before § 243. According to that §, however, if we change x into $y+1$, we get $2y^2+4y+4$: and by assuming this equal to $(2-vy)^2$, and resolving the equation, we obtain $y=\frac{4v+4}{v^2-2}$, and thence $x=\frac{4(v+1)}{v^2-2}+1$. Hence, if $v=2, x=7$; if $v=1\frac{1}{2}$, $x=41$, &c.*

7. What value of y will make the value of x rational in the equation, $ax^2+bx+cy^2=0$?

By resolving this equation for x, we find that its value contains the radical $\sqrt{(b^2-4acy^2)}$. To render this rational, assume it equal to $b-vy$. Then, by squaring, and by resolving the result, we get $y=\frac{2bv}{v^2+4ac}$. Thus, for example, if $a=1$, $b=2$, and $c=-3$, and if we assume $v=4$, we get $y=4$, and $x=-1\pm7$. The general values of x are readily found to be $-\frac{4bc}{v^2+4ac}$, and $-\frac{b}{a}\cdot\frac{v^2}{v^2+4ac}$.

8. Find n numbers, such that if p_1 be added to the first, p_2 to the second, p_3 to the third, &c., p_1, p_2, &c., being given numbers, positive or negative, the sums may all be squares, and such that the sum of all the required numbers may be a square.

Here, by assuming $a_1^2-p_1$, $a_2^2-p_2$, $a_3^2-p_3$, ..., x^2-p_n, where a_1, a_2, &c., are any determinate numbers assumed at pleasure, we satisfy all the conditions of the question except the last.

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* To generalise this so as to make $2x^2+2a^2$ a square, we have merely to change x into $a^{-1}x$. Then, if we multiply by a, the final result obtained above becomes $x=\frac{4a(v+1)}{v^2-2}+a$.

To satisfy this, we add all the assumed quantities together, thus obtaining

$$x^2+a_1^2+a_2^2+\ldots+a_{n-1}^2-p_1-p_2-\ldots-p_n;$$

an expression which will be rendered a square by assuming it (§ 239.) equal to $(x+v)^2$.

9. To find three numbers in arithmetical progression, such that the sum of each pair of them may be a square.

This question may be solved very simply in the following manner. Assume the sums of the first and second, and of the first and third, of the required numbers, respectively equal to the squares, $x^2-2xy+y^2$, and $x^2+2xy+y^2$. Then, $4xy$, the difference of these, is plainly the difference of the second and third, and consequently of the first and second. Hence (§ 52.) we find the first, second, and third to be

$\frac{1}{2}x^2-3xy+\frac{1}{2}y^2$, $\frac{1}{2}x^2+xy+\frac{1}{2}y^2$, and $\frac{1}{2}x^2+5xy+\frac{1}{2}y^2$, respectively; and it now remains only to make $x^2+6xy+y^2$, the sum of the second and third, a square. To effect this, assume it equal to $(x+v)^2$: then, rejecting x^2, and resolving the equation, we get

$x=-\dfrac{v^2-y^2}{2(v-3y)}$ [*]; where v and y may be assumed at pleasure, except that v cannot be equal to $\pm y$ nor to $3y$. As an example, let $y=1$ and $v=19$. Then, $x=-\frac{45}{4}$, and the required numbers will be found to be $\frac{3121}{32}$, $\frac{1681}{32}$, and $\frac{241}{32}$, which will answer. By multiplying these, also, by the square number 64, we get the integers, 6242, 3362, and 482, which will be found to answer equally.

Exercises.

1. Find a number, such that the sum of its square and cube may be a square.

Ans. v^2-1, where v may be any number whatever.

2. Find two numbers, such that if to each of them, and to

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[*] By taking, for simplicity, $y=1$ in this value of x, and substituting the result in the expressions found above for the required numbers; and then, by multiplying the quantities so obtained by $(v-3)^2$, we find that the three required numbers will be $x'+3x''$, $x'-x''$, and $x'-5x''$, where $x'=\frac{1}{8}(v^2-1)^2+\frac{1}{2}(v-3)^2$, and $x''=\frac{1}{2}(v-3)(v^2-1)$; and, if v be an odd number, these results will be whole numbers; while, if v be 17 or more, they will be positive. Thus, if $v=17$, the three numbers will be 16514, 8450, and 386; and if $v=21$, they will be 36242, 20402, and 4562.

their sum and difference, 1 be added, each of the four sums may be a square[*] *Ans.* 168 and 120.

3. Find two numbers, such that the difference of their cubes may be a square number. *Ans.* $\frac{2v+3}{v^2-3}$; and $\frac{v^2+2v}{v^2-3}$.

4. Find two numbers, such that the difference of their squares may be a cube, and the difference of their cubes a square.

Ans. $10v^6$ and $6v^6$.[†]

5. Render $2x^2-2$ a square.[‡]

Ans. $x=\frac{v^2+2}{v^2-2}$, and $2x^2-2=\left(\frac{4v}{v^2-2}\right)^2$.

6. What value of x will make $13x^2+15x+7$ a square?

Ans. $x=\frac{3v^2-26v+54}{v^2-13}$; particular values of x, $\frac{6}{23}$, $\frac{19}{36}$, &c.

7. Find two numbers, such that if each be added to the square of the other, the sum shall be a square.

Ans. $\frac{4(v^2+1)}{8v+1}$, and $\frac{v^2-8v}{8v+1}$.[§]

8. Find two numbers, such that their sum and difference shall be squares.

Ans. v^2+1 and $2v$ [‖]; particular values 5 and 4, 10 and 6, 17 and 8, &c.

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[*] To solve this question, assume the numbers equal to x^2+2x and x^2-2x. Then $4x+1$ and $2x^2+1$ must be squares. Assume the former equal to v^2, and in the second take x equal to the value of it so obtained. The expression thus found will be a square, when $v=1$, and the solution will be completed by means of §§ 243 and 244.

[†] These answers may be obtained by assuming the numbers equal to x^3+2a^6 and x^3-2a^6.

[‡] Here take $x=y+1$, and the solution will be easy. By taking $v=2$, we get $x=3$; also, $v=\frac{3}{2}$ gives $x=17$; while if $v=\frac{10}{7}$, we get $x=99$.

[§] These expressions will be obtained by assuming $x-1$ and $4x$ as the required numbers.

[‖] These values are assumed at once, in accordance with § 53. It would be, in appearance at least, more general to take instead of them $v^2+v'^2$ and $2vv'$. Bonnycastle, in an inelegant solution, assumes as the required numbers, x and x^2+x; and, after the trouble of a formal investigation, obtains for the numbers expressions in a considerable degree complicated. The next question is also from Bonnycastle. In solving it, he assumes $4x$, x^2-4x, and $2x+1$, as the required numbers, and he thus obtains very simply the answers here given.

9. Required three numbers, such that the sum of all three, and the sum of every pair of them, may be square numbers.

Ans. $\frac{2}{3}(v^2-1)$, $\frac{1}{36}(v^2-1)^2-\frac{2}{3}(v^2-1)$, and $\frac{1}{3}(v^2+2)$.

10. Render $6x^3+9x^2+36x-8$ a cube.

Ans. It will be a cube when $x = 3$.

11. What value of x will make $3x^4-10x^3+24x^2+32x+16$ a square? *Ans.* 9.

12. Find two numbers, such that their sum and the sum of their cubes may be equal.

Ans. 1 and 1; $\frac{3}{7}$ and $\frac{8}{7}$; $\frac{5}{7}$ and $\frac{8}{7}$; $\frac{8}{13}$ and $\frac{15}{13}$, &c.; general values, $\frac{4v}{v^2+3}$ and $\frac{2v\pm(v^2-3)}{v^2+3}$.

13. Find a number, such, that if 1 be added to its double and triple, each of the results may be a square.

Ans. 40, 3960, $\frac{48}{25}$, &c.; general value $2(x^2+x)$, where $x=\frac{2v+6}{v^2-6}$.

14. Resolve 85, which is the sum of 2^2 and 9^2, into two other squares.

Ans. 6^2 and 7^2, besides innumerable fractional answers.

15. Find two numbers, such that their difference may be equal to the difference of their squares, and that the sum of their squares may be a square.

Ans. $\frac{2v-2}{v^2-2}$, and $1-\frac{2v-2}{v^2-2}$ or $\frac{v^2-2v}{v^2-2}$.*

16. To divide a number which is the sum of three square numbers in arithmetical progression, into three other squares which shall also be in arithmetical progression.

Ans. The numbers will be found by dividing one third of the given number into two squares, a^2 and b^2, by Exam. 3., and taking $(a-b)^2$, a^2+b^2, and $(a+b)^2$ as the required numbers.†

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* This question is taken from Barlow's *Theory of Numbers*, p. 477., where it is stated that the answer is "$\frac{4}{7}$, $\frac{3}{7}$, or any two fractions the sum of which is unity." This is erroneous, as, besides having their sum equal to unity, the fractions must be of the forms given above, or some equivalent ones.

† As a particular example, let us take 1, 25, and 49, which are square numbers in arithmetical progression. The sum of these is 75, and one third of this, 25, is the sum of the squares of 4 and 3 (half the sum and half the difference of $\sqrt{49}$ and $\sqrt{1}$). Then, by taking $a = 4$, $b = 3$, and $v = 3$, in the result obtained in Exam. 3., we find two other squares whose

CHAPTER XV.

SERIES.

251. **The** series which are treated of in mathematics are successions of quantities, each of which, after a certain term, is derived, according to a determinate law, from one or more of the terms preceding it. Thus, in the series, $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{18}$, $\frac{1}{54}$, &c., each term is one third of the one next before it; while, in the series, 1, 2, 3, 5, 8, 13, &c., each term after the second is the sum of the two immediately preceding ones. Out of the numberless kinds of such series, two have already been treated of in Chapter VII.; and we may now briefly consider a few others.

252. A principal problem regarding series is the finding of their sums. If an infinite series, that is, a series to the number of whose terms there is no limit, be proposed, and if we take as its sum the sum of a certain number of its leading terms, the first ten or first hundred, for example, the error would evidently be the amount of all the remaining terms. Now, if we can show, that by taking more and more of the terms, that error, or complement, admits of being rendered as small as we please, — less, in fact, than any number that can be assigned, however small, it is plain that there is a limit to which, whether we can find it or not, the sum of the series tends as its value. Such a series is said to be *convergent*, and all others to be *divergent*. Thus, we saw in page 112., that the sum of the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., can never amount to 1, however far we may carry it; but that, if we take a sufficient number of terms, their sum will differ from 1 by as small a quantity as we please: this series therefore is convergent. On the contrary, the sum of the series, 1, 1, 1, &c., and 1, 2, 3, 5, 8, &c., will evidently go on perpetually in-

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sum is 25, to be $(\frac{24}{5})^2$ and $(\frac{7}{5})^2$, and the required numbers will be $(\frac{17}{5})^2$, 5^2, and $(\frac{31}{5})^2$; or, by multiplying by the square 25, 289, 625, and 961, which are found to answer. Innumerable other answers may be found by giving different values to v.

creasing, as more terms are taken, and will surpass any limit however great: these, therefore, are diverging series.*

253. It is impossible to give a general rule for the summation of series. Certain classes, however, may be readily summed; and we may now consider some of these, commencing with the following simple example.

Exam. 1. Find the sum of the series, $\frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \frac{1}{4.5}$, &c.†

Here the first term is plainly equal to $\frac{1}{1}-\frac{1}{2}$, the second to $\frac{1}{2}-\frac{1}{3}$, the next to $\frac{1}{3}-\frac{1}{4}$, and the nth or general term, which is evidently $\frac{1}{n(n+1)}$, to $\frac{1}{n}-\frac{1}{n+1}$. Hence, therefore, denoting the

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* Such also is the series, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, &c., already referred to in p. 113., though the contrary might be supposed from the continual diminution of its terms, which, by carrying it out far enough, may be made as small as we please. To show this, let the sum of the series be written thus:

$$\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+(\tfrac{1}{9}+\tfrac{1}{10}+\ \ldots\ +\tfrac{1}{16})+\&c.$$

We thus see, that, while the first term is $\frac{1}{2}$, the sum of the next two, being greater than $\frac{1}{4}+\frac{1}{4}$, is greater than $\frac{1}{2}$; and that the sum of the next four, being greater than $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$, is also greater than $\frac{1}{2}$. In a similar way it would appear, that the sum of the next eight terms, the sum of the sixteen following them, and so on, would be each greater than $\frac{1}{2}$; and it would be shown in a similar manner, that, universally, if $2v$ be the denominator of any term after the first, the sum of the v terms ending with that one would exceed $\frac{1}{2}$. Hence the sum of the infinite series must be infinite, being greater than $\frac{1}{2}$ repeated an infinite number of times.

† This series may be generated by dividing, as in the margin, 1 by 1, under the form $2-1$; the remainder $\frac{1}{2}$ by 1 under the form $3-2$; the next remainder $\frac{1}{3}$ by $4-3$; and so on. In a similar manner, the student may evolve as many series as he may think proper, by subjecting the divisor to a regular change of form: and the sum of an *infinite* number of terms of each will be the exact quotient obtained by dividing the original dividend by the divisor. Thus, for instance, by dividing 1 by 4 under the successive forms, $5-1$, $7-3$, $9-5$, &c., we should get $\frac{1}{5}+\frac{1}{5.7}+\frac{3}{5.7.9}+\frac{3}{7.9.11}+$ &c.; in which $S_\infty=\frac{1}{4}$. If all the terms be divided by 3, this will take the form, $\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+$ &c.; and in this we shall plainly have $S_\infty=\frac{1}{12}$.

$$\begin{array}{l} 2-1)\,1\ \left(\frac{1}{1.2}+\frac{1}{2.3}+\&c.\right. \\ \qquad\ \underline{1-\frac{1}{2}} \\ 3-2)\,\frac{1}{2} \\ \qquad\quad \underline{\frac{1}{2}-\frac{1}{3}} \\ \qquad\qquad\ \frac{1}{3},\ \&c. \end{array}$$

sum of n terms by S_n, and writing the positive terms in one line, and the negative in another, we get

$$S_n=\left\{\begin{array}{l}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\,..+\frac{1}{n}\\ \quad-\frac{1}{2}-\frac{1}{3}-\,..-\frac{1}{n}-\frac{1}{n+1}\end{array}\right\}, \text{ or } S_n=1-\frac{1}{n+1}=\frac{n}{n+1}.$$

Hence the sum of six terms is $\frac{6}{7}$, and the sum of a thousand is $\frac{1000}{1001}$. If n be infinite the fractional part of the first form of the sum vanishes, and we have simply $S_\infty=1$.

254. To generalise this process, since

$$\frac{1}{p}-\frac{1}{p+q}=\frac{q}{p(p+q)}, \text{ we have, conversely, } \frac{1}{p(p+q)}=\frac{1}{q}\left(\frac{1}{p}-\frac{1}{p+q}\right).$$

By means of this expression, we can find two fractions, whose difference shall be equal to any given fraction; and thus we can determine the sum of any series which admits of being summed on this principle.

Exam. 2. Find the sum of the series, $\frac{1}{3.5}, \frac{1}{4.6}, \frac{1}{5.7}, \frac{1}{6.8}$, &c.

Here the general term is evidently

$$\frac{1}{(n+2)(n+4)}, \quad \text{or} \quad \frac{1}{2}\left(\frac{1}{n+2}-\frac{1}{n+4}\right),$$

by what has just been shown, p being equal to $n+2$, and q to 2. Hence, by taking n successively equal to 1, 2, 3, &c., we find the first term, the second, the third, &c., to be

$$\tfrac{1}{2}(\tfrac{1}{3}-\tfrac{1}{5}),\ \tfrac{1}{2}(\tfrac{1}{4}-\tfrac{1}{6}),\ \tfrac{1}{2}(\tfrac{1}{5}-\tfrac{1}{7}), \text{ \&c.};$$

and the term before the last, the term before that, &c. will be resolved into their parts by changing n into $n-1$, $n-2$, &c. Hence,

$$S_n=\frac{1}{2}\left\{\begin{array}{l}\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{n+2}\\ \qquad\quad-\frac{1}{5}-\frac{1}{6}-\cdots-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}\end{array}\right\};$$

$$\text{or } S_n=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{n+3}-\frac{1}{n+4}\right)=\frac{7}{24}-\frac{2n+7}{2(n+3)(n+4)}.$$

If n be infinite, the third and fourth terms of the sum in its first form will vanish, so that the sum of the infinite series is $\frac{7}{24}$.

255. Since $\dfrac{1}{p_1p_2p_3\cdots p_{n-1}}-\dfrac{1}{p_2p_3p_4\cdots p_n}=\dfrac{p_n-p_1}{p_1p_2p_3\cdots p_n}$,[*] by dividing by p_n-p_1, we get

$$\frac{1}{p_1p_2p_3\cdots p_n}=\frac{1}{p_n-p_1}\left(\frac{1}{p_1p_2p_3\cdots p_{n-1}}-\frac{1}{p_2p_3p_4\cdots p_n}\right).$$

This very general formula enables us to sum many series, in which the factors of each of the denominators are equidifferent, the common difference being the same in all the denominators. The following examples will illustrate this method.

Exam. 3. Find the sum of the series,

$$\frac{1}{1.4.7},\ \frac{1}{4.7.10},\ \frac{1}{7.10.13},\ \&c.$$

Here, the factors of each denominator exceeding those of the preceding one by 3, it is easy to see that the general term is $\dfrac{1}{(3n-2)(3n+1)(3n+4)}$, since, if n be taken successively equal to 1, 2, 3, &c., that expression will give the several terms of the series. Hence, by taking $p_1=3n-2$, $p_2=3n+1$, and $p_n=3n+4$, we find by what was shown above, that the general term is equivalent to $\dfrac{1}{6}\left\{\dfrac{1}{(3n-2)(3n+1)}-\dfrac{1}{(3n+1)(3n+4)}\right\}$.

Taking in this n successively equal to 1, 2, 3, ..., and $n-1$, we find the first, second, third, and $(n-1)$th terms to be $\dfrac{1}{6}\left(\dfrac{1}{1.4}-\dfrac{1}{4.7}\right)$, $\dfrac{1}{6}\left(\dfrac{1}{4.7}-\dfrac{1}{7.10}\right)$, $\dfrac{1}{6}\left(\dfrac{1}{7.10}-\dfrac{1}{10.13}\right)$, and $\dfrac{1}{6}\left\{\dfrac{1}{(3n-5)(3n-2)}-\dfrac{1}{(3n-2)(3n+1)}\right\}$. Then, by arranging these and the nth term as in the other examples, we get

$$S_n=\frac{1}{6}\left\{\begin{array}{l}\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdot\cdot+\frac{1}{(3n-5)(3n-2)}\dagger+\frac{1}{(3n-2)(3n+1)}\\ \quad-\frac{1}{4.7}-\frac{1}{7.10}-\cdot\cdot-\cdots\cdots\cdots\cdots-\frac{1}{(3n-2)(3n+1)}-\frac{1}{(3n+1)(3n+4)}\end{array}\right\}:$$

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[*] Found by multiplying the numerator and denominator of the first fraction by p_n, and those of the second by p_1, and then subtracting the second result from the first.

[†] This term would be destroyed by one of the parts of the $(n-2)$th term.

$$\text{or } S_n=\frac{1}{6}\left\{\frac{1}{1.4}-\frac{1}{(3n+1)(3n+4)}\right\}.$$

When n is infinite, the last term vanishes, and therefore the sum of an infinite number of terms is $\frac{1}{24}$.

Exam. 4. Sum the series, $\frac{5}{1.2.3}, \frac{6}{2.3.4}, \frac{7}{3.4.5}, \frac{8}{4.5.6}$, &c.

Here the general term is plainly $\frac{n+4}{n(n+1)(n+2)}$; and this, by the last §, is equivalent to $\frac{1}{2}\left\{\frac{n+4}{n(n+1)}-\frac{n+4}{(n+1)(n+2)}\right\}$.

Then, by taking n successively equal to 1, 2, 3, ..., n, and arranging the results in the usual way, we get

$$S_n=\frac{1}{2}\left\{\begin{array}{l}\frac{5}{1.2}+\frac{6}{2.3}+\frac{7}{3.4}+\ldots+\frac{n+3}{(n-1)n}+\frac{n+4}{n(n+1)} \\ \qquad -\frac{5}{2.3}-\frac{6}{3.4}-\ldots-\ldots\ldots-\frac{n+3}{n(n+1)}-\frac{n+4}{(n+1)(n+2)}\end{array}\right\}$$

$$=\frac{1}{2}\left\{\frac{5}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}-\frac{n+4}{(n+1)(n+2)}\right\},$$

by contraction. Now the first term is the half of $2\frac{1}{2}$; so that if we omit the 2 and the last term, what remains is one half of the series in Exam. 1. Using, therefore, instead of this series, the sum there found, we get

$$S_n=\frac{1}{2}\left\{2+1-\frac{1}{n+1}-\frac{n+4}{(n+1)(n+2)}\right\}=\frac{3}{2}-\frac{n+3}{(n+1)(n+2)}.$$

Hence $S_\infty=\frac{3}{2}$.

Exercises. Find the sums of the following series.

1. $\frac{1}{4.5}+\frac{1}{5.6}+\frac{1}{6.7}+\frac{1}{7.8}+$&c.

Ans. $S_n=\frac{1}{4}-\frac{1}{n+4}=\frac{n}{4(n+4)}$, and $S_\infty=\frac{1}{4}$

2. $\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+$&c.

Ans. $S_n=\frac{1}{12}-\frac{1}{2}\cdot\frac{1}{(n+2)(n+3)}$, and $S_\infty=\frac{1}{12}$.

3. $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\&c.$

Ans. $S_n=\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$, and $S_\infty=\frac{1}{18}$.

4. $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\&c.$

Ans. $S_n=\frac{3}{4}-\frac{2n+3}{2n^2+6n+4}$, and $S_\infty=\frac{3}{4}$.

5. $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\&c.$

Ans. $S_n=\frac{11}{18}-\frac{1}{3}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)$, and $S_\infty=\frac{11}{18}$.

6. $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+\&c.$

Ans. $S_n=\frac{1}{12}-\frac{1}{4(2n+3)}.(-1)^n$, and $S_\infty=\frac{1}{12}$.

256. Infinite series may be converted into continued fractions; and thus their sums may be determined, either accurately or approximately, in many of the most interesting cases.* The method of effecting this will be illustrated by the following examples, in which the series is regarded as the numerator of a fraction having unity as its denominator.

Exam. 5. Express the infinite series, $1-4x+7x^2-10x^3+13x^4-\&c.$, as a continued fraction.

Here the operations in division for obtaining the continued fraction will stand very conveniently in the following form, the coefficients only being used, and, as in the note in p. 61., the French and the common mode of placing the divisor being employed alternately.

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* Much of what follows, regarding this curious and interesting application of continued fractions, is taken in substance from Burg's *Lehrbuch der höhern Mathematik* (Wien, 1832), vol. i. chap. 10. See also Ettingshausen's *Vorlesungen über die höhere Mathematik* (Wien, 1827), p. 69. &c. When, by the method here employed, or by any other, an exact finite quantity is obtained as the *sum* of an infinite series, the meaning is, that, if the quantity so found be developed by the performance of the operations indicated, it will become the same as the proposed series.

$$\begin{array}{rrrrr|rrrrr|l}
1 & & & & & 1 & -4 & 7 & -10 & 13- & 1=q_1 \\
1 & -4 & 7 & -10 & 13- & 1 & -\frac{7}{4} & \frac{10}{4} & -\frac{13}{4} & \frac{16}{4}- & \frac{1}{4x}=q_2 \\
\cline{1-10}
 & 4 & -7 & 10 & -13+ & & -\frac{9}{4} & \frac{18}{4} & -\frac{27}{4} & \frac{36}{4}- & -\frac{16}{9}=q_3 \\
 & 4 & -8 & 12 & -16+ & & -\frac{9}{4} & \frac{18}{4} & -\frac{27}{4} & \frac{36}{4}- & -\frac{9}{4x}=q_4.^* \\
\cline{2-5}\cline{7-10}
 & & 1 & -2 & 3- & & & 0 & & &
\end{array}$$

Hence $1-4x+7x^2-$ &c.

$$=\frac{1}{1}+\cfrac{1}{\left(\frac{1}{4x}\right)+\cfrac{1}{\left(\frac{-16}{9}\right)+\cfrac{1}{\left(\frac{-9}{4x}\right)}}} \qquad =\frac{1}{1}+\cfrac{4x}{1-\cfrac{9x}{4+\cfrac{x}{1}}}.\dagger$$

257. The value of the continued fraction found above, in its first form, will be obtained in the manner pointed out in § 216., and the work will stand as follows:

---

* In the process given above, 1 in the left-hand division of the work is divided by 1 −4 7, &c., in the right-hand division; and the quotient q_1 is 1, and the remainder 4 −7 10, &c. Dividing 1 −4 7, &c., by this remainder, we get the next quotient $q_2=\frac{1}{4x}$, with the remainder $-\frac{9}{4}$ $\frac{18}{4}$ $-\frac{27}{4}$, &c. We next divide the last remainder, 4 −7, &c., by this, and we get as quotient $-\frac{16}{9}$, and as remainder 1 −2 3, &c. By this, in the last place, we divide $-\frac{9}{4}$ $\frac{18}{4}$, &c.; and we thus obtain the quotient q_4, with no remainder.

† The first form of this continued fraction is that which is obtained according to the ordinary process in Chap. XII. To obtain the second form, we multiply the numerator and denominator of what is annexed to 1, the first denominator, by the denominator $4x$. In this way, $\frac{4x}{1}$, the second fraction, is found; and the numerator of the third becomes $4x$ instead of 1. We then multiply this numerator and its denominator by the denominator 9; and we thus get for numerator $36x$, and for denominator -16, while the numerator of the remaining fraction becomes 9: and, by multiplying the terms of this fraction by the denominator $4x$, they are changed into $36x$ and -9, and the fraction itself becomes $-4x$. Then we simplify the last two fractions by dividing by 4; and finally by changing some signs, the method of doing which is quite obvious, we get the fraction in the second form.

$$1,\quad \frac{1}{4x},\quad -\frac{16}{9},\quad -\frac{9}{4x}$$

$$\frac{1}{1},\quad \frac{\left(\frac{1}{4x}\right)}{1+\frac{1}{4x}},\quad \frac{1-\frac{4}{9x}}{1-\frac{16}{9}-\frac{4}{9x}},\quad \frac{\frac{1}{4x}-\frac{9}{4x}+\frac{1}{x^2}}{1+\frac{1}{4x}-\frac{9}{4x}+\frac{4}{x}+\frac{1}{x^2}}=\frac{1-2x}{1+2x+x^2}.$$

This is the required sum; and its correctness would be verified by actually dividing the numerator by the denominator, as the quotient would be the given series.* It may be remarked that, in the present example and in similar ones, the sum would be found rather more simply from the second form of the continued fraction, by reducing the last two of its component fractions to a simple fraction; then by attaching the result to the denominator of the preceding one, and reducing the complex fraction so obtained to a simple one, and so on.

Exam. 6. Given $1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\&c.$ †; to reduce it to a continued fraction.

Here, by dividing 1 by the given series, the series by the remainder, and so on in the usual way, we find the successive quotients to be 1, $-\frac{1}{x}$, -2, $\frac{3}{x}$, 2, $-\frac{5}{x}$, -2, $\frac{7}{x}$, &c.; where the law of continuation is manifest. Hence the continued fraction will be of either of the following forms, the second being derived from the first in the manner pointed out in the note to the last example:

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* A series, such as the one in this example, which can be expressed by a finite continued fraction, and for which, therefore, a sum, or generating fraction, can be found, which by division will reproduce the series, is called a *recurring series*. In a series of this kind, each term, after a certain number at the commencement, is equal to the sum obtained by adding together the respective products of a certain number of the terms immediately preceding it, by determinate multipliers. Thus, in the present example, each term after the second is obtained by multiplying the two terms immediately preceding it by $-x^2$ and $-2x$ respectively, and taking the sum of the results; as would appear by dividing the numerator $1-2x$, by the denominator $1+2x+x^2$, of the generating fraction. These multipliers (in the present case $-x^2$ and $-2x$) have been called the *scale of relation* of the series. In summing series by means of continued fractions, it is not necessary to consider this scale.

† This is the series equivalent to the exponential function e^x, e being the base of the Neperian logarithms. See *Differential and Integral Calculus*, p. 33.

$$\cfrac{1}{1+\cfrac{1}{\left(\frac{-1}{x}\right)+\cfrac{1}{-2+\cfrac{1}{\left(\frac{3}{x}\right)+\cfrac{1}{2+\cfrac{1}{\left(\frac{-5}{x}\right)+\&c.}}}}}};$$

or,
$$\cfrac{1}{1-\cfrac{x}{1+\cfrac{x}{2-\cfrac{x}{3+\cfrac{x}{2-\cfrac{x}{5+\cfrac{x}{2-\cfrac{x}{7+\&c.}}}}}}}}\text{*}$$

Exam. 7. Reduce $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\&c.$† to a continued fraction.

Here, by dividing 1 by the series, the series by the remainder, &c., we find the successive quotients to be

$$\frac{1}{x},\ \frac{2}{1},\ \frac{3}{x},\ \frac{2}{2},\ \frac{5}{x},\ \frac{2}{3},\ \frac{7}{x},\ \frac{2}{4},\ \&c.;$$

the law of continuation of which is evident. Hence we shall have the equivalent continued fraction expressed in either of the following forms, the latter being derived from the former in the way pointed out in the note to Exam. 1.:

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* It would be easy to show by the method pursued in § 216., that, in finding the converging fractions, when there is a negative quotient, the products by it are to be severally *diminished*, and not *increased*, by the terms of the fraction immediately preceding them. Hence, if in this fraction we take $x = 1$, we shall find the first twelve of the fractions converging to the value of ϵ, in the following manner:

$$\begin{array}{ccccccccccccc} 0, & 1, & -1, & 2, & -3, & 2, & -5 & 2 & -7 & 2 & -9 & 2 & -11. \\ \frac{0}{1}, & \frac{1}{1}, & \frac{1}{0}, & \frac{2}{1}, & \frac{8}{3}, & \frac{19}{7}, & \frac{87}{32}, & \frac{193}{71}, & \frac{1264}{465}, & \frac{2721}{1001}, & \frac{23225}{8544}, & \frac{49171}{18089}, & \frac{517656}{190435}. \end{array}$$

By performing the actual division in the last of these fractions, we get 2·718281828445; a result which is true in all its figures except the last two, which ought to be 59.

† This (*Differential and Integral Calculus*, p. 34.) is the Neperian logarithm of $1+x$.

$$\cfrac{1}{\left(\frac{1}{x}\right)+\cfrac{1}{\left(\frac{2}{1}\right)+\cfrac{1}{\left(\frac{3}{x}\right)+\cfrac{1}{\left(\frac{2}{2}\right)+\cfrac{1}{\left(\frac{5}{x}\right)+\&c.;}}}}}$$

or, $$\cfrac{x}{1+\cfrac{x}{2+\cfrac{x}{3+\cfrac{2x}{2+\cfrac{2x}{5+\cfrac{3x}{2+\cfrac{3x}{7+\cfrac{4x}{2+\cfrac{4x}{9+\&c.}}}}}}}}}*$$

Exercises.

7. Show how $1+2x+3x^2+4x^3+\&c.$ may be expressed as a continued fraction; and thence find the sum of the series.

Ans. The several quotients are 1, $-\frac{1}{2x}$, -4, and $\frac{1}{2x}$. In the second form of the continued fraction, the component fractions are $\frac{1}{1}$, $-\frac{2x}{1}$, $\frac{2x}{4}$, and $-\frac{2x}{1}$; and the sum is $\frac{1}{(1-x)^2}$.

8. Find the sum of the infinite series,

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* If, in either of these expressions, x be assumed equal to 1, and if eleven of the component fractions of the continued one be taken, the equivalent common fraction will be $\frac{1823088}{2630160}$, or, by division, 0·6931471849, which is the Neperian logarithm of 2, true in all its figures except the last two. Now, it is worthy of remark, that if x were taken equal to 1 in the series given in the example, the resulting series, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\&c.$, would converge so slowly, that it would be necessary to employ thousands of millions of its terms to give a result of equal accuracy, and that the labour of performing the computation in that way would be almost absolutely insuperable. We have thus a striking instance of the advantage obtained by employing continued fractions in cases of this kind.

$$x+x^2-x^4-x^5+x^7+x^8-x^{10}-x^{11}+\&c.,$$

by means of a continued fraction.

Ans. $\dfrac{x}{1-x+x^2}$.

9. The sine of a circular arc x to the radius 1 is

$$x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2....5}-\frac{x^7}{1.2....7}+\&c.,$$

and its cosine $1-\dfrac{x^2}{1.2}+\dfrac{x^4}{1.2.3.4}-\dfrac{x^6}{1.2....6}+\&c.$; and if the former be divided by the latter, the quotient is the tangent of x.* Show, from this, that the tangent of x is equivalent to the continued fraction,

$$\cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{x^2}{7-\&c.}}}}$$

10. Prove that $(1+x)^n$ is equivalent to the continued fraction in which the numerators of its constituent fractions are 1, nx, $\frac{1}{2}(n+1)x$, $\frac{1}{6}(n-1)x$, $\frac{1}{6}(n+2)x$, $\frac{1}{10}(n-2)x$, $\frac{1}{10}(n+3)x$, &c.; and the denominators each 1; the signs of the several fractions being + and − alternately.

11. By employing the first five terms of the continued fraction found in Exam. 6., show that the series from which it arises is nearly equal to $\dfrac{12+6x+x^2}{12-6x+x^2}$.

12. By means of the first six terms of the continued fraction found in Exam. 7., show that the series from which it was obtained, is nearly equal to $\dfrac{60x+60x^2+11x^3}{60+90x+36x^2+3x^3}$.

13. Required the generating fraction which produces the series,

$\dfrac{1}{x}-\dfrac{3}{x^2}+\dfrac{5}{x^3}-\dfrac{7}{x^4}+\dfrac{9}{x^5}-\&c.$ *Ans.* $\dfrac{x-1}{(x+1)^2}$.

14. Find the sum of the infinite series, $1^3x+2^3x^2+3^3x^3+4^3x^4$ $+5^3x^5+\&c.$ *Ans.* $\dfrac{x+4x^2+x^3}{(1-x)^4}$.

258. The sums of series may sometimes be readily found by

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* See *Differential and Integral Calculus*, pp. 39 and 40.

the method of indeterminate coefficients. This will be illustrated by the following example.

Exam. 8. Required the sum of $1^2, 2^2, 3^2, \ldots, n^2$.

Here let the required sum be assumed equal to $An+Bn^2+Cn^3$.[*] Then, by changing n into $n+1$, we get $1^2+2^2+\ldots\ldots+n^2+(n+1)^2=A(n+1)+B(n+1)^2+C(n+1)^3$. From this take $1^2+2^2+\ldots\ldots\ldots+n^2=An+Bn^2+Cn^3$, and there will remain $(n+1)^2$, or $n^2+2n+1=A+2nB+B+3Cn^2+3Cn+C$. Hence, by equalling the coefficients of the like powers of n we get $3C=1$, $2B+3C=2$, and $A+B+C=1$; and, by resolving these equations, we find $C=\frac{1}{3}$, $B=\frac{1}{2}$, and $A=\frac{1}{6}$. The required sum, therefore, is $\frac{1}{6}n+\frac{1}{2}n^2+\frac{1}{3}n^3$, or, what is equivalent, $\frac{n(n+1)(2n+1)}{1.2.3}$.[†]

Exercises. Find the sums of the following series.

15. $1^3, 2^3, 3^3, \ldots, n^3$.	*Ans.* $\{\frac{1}{2}n(n+1)\}^2$.[‡]
16. $1^4, 2^4, 3^4, \ldots, n^4$.	*Ans.* $\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$.
17. $1^5, 2^5, 3^5, \ldots, n^5$.	*Ans.* $\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2$.
18. $1^2, 3^2, 5^2, \ldots, (2n-1)^2$.	*Ans.* $\frac{1}{3}n(4n^2-1)$.
19. $1.2, 2.3, 3.4, \ldots, n(n+1)$.	*Ans.* $\frac{1}{3}(2n+3n^2+n^3)$.
20. $1.2^2, 2.3^2, 3.4^2, \ldots, n(n+1)^2$.	*Ans.* $\frac{5}{6}n+\frac{7}{4}n^2+\frac{7}{6}n^3+\frac{1}{4}n^4$.
21. $1.3^2, 3.5^2, 5.7^2, \ldots, (2n-1)(2n+1)^2$.	*Ans.* $n^2(2n^2+3)+\frac{1}{3}n(16n^2-4)$.

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[*] It is evident, that $1^2+2^2+3^2+\ldots+n^2$ is less than $n^2+n^2+n^2+$, &c., carried out to n terms, that is, than n^3. At the same time, as there are n terms, and as one of them is n^2, it is plain that n^3 will enter in some way into the sum; and hence the reason of assuming it equal to $An+Bn^2+Cn^3$. It would be found, indeed, that if we had assumed it equal to $A'+An+Bn^2+Cn^3+Dn^4+En^5+$&c., A' would disappear in the operation, and each of the coefficients after C would be found to be nothing. It is plain, also, that there can be no term A' not containing n, from the consideration, that when n is nothing, the sum must be nothing, and not A'. Since, also, the terms are all whole numbers, it is evident that the sum can contain no powers of n having fractional or negative indices. These remarks are applicable in all similar cases.

[†] Much additional and important information regarding series will be found in the author's *Treatise on the Differential and Integral Calculus*, Sect. XXIV.

[‡] By § 133. the sum of $1, 2, 3, \ldots, n$ is $\frac{1}{2}n(n+1)$. Hence we arrive at the curious conclusion, that

$$(1+2+3+\ldots\ldots+n)^2=1^3+2^3+3^3+\ldots\ldots+n^3.$$

CHAPTER XVI.

APPLICATION OF ALGEBRA IN INVESTIGATIONS IN GEOMETRY.*

Exam. 1. GIVEN the perimeter of a right-angled isosceles triangle $= p$; to find its sides.

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* The solutions of two geometrical questions by means of algebra have been already given in Chapter IX. In the present section, the mode of resolving geometrical problems by this means is taken up briefly in a separate form.

The chief principles necessary for the solutions of the following examples and exercises are as follows; and their demonstrations will be found in the author's other works, and in his edition of *Euclid* as referred to.

1. The area of a parallelogram is computed by multiplying its length by its perpendicular breadth; that of a square by multiplying a side by itself; and that of a triangle by multiplying its base by its perpendicular, and taking half the product. (*Euc.* I. 46. Corollaries.)

2. If we put π to denote 3·14159265, &c. (half the circumference of the circle whose radius is 1), the area of the circle whose radius is a is πa^2. The volume, or solid content, of the sphere whose radius is a is $\frac{4}{3}\pi a^3$, and the area of its surface is $4\pi a^2$. (*Differential and Integral Calculus*, Section XX.)

3. The volume of a cone or pyramid is one third of the product obtained by multiplying the area of its base by its perpendicular height. (*Euc.* XII. 7. Cor. and 10.; and *Diff. and Int. Calc.*, Sect. XX.)

4. If a be the radius of the base of a cone, and b its slant height, the area of its curve surface will be $\frac{1}{2}\pi ab$. This is manifest from supposing the surface developed into a sector of a circle.

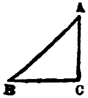
5. In a right-angled triangle, the square described on the hypotenuse (the side opposite to the right angle) is equal to the squares described on the legs (the other sides). (*Euc.* I. 47.)

6. In triangles which are equiangular to one another, the sides which are opposite to equal angles are proportional. (*Euc.* VI. 4.)

7. If from a point without a circle two straight lines be drawn cutting it, and be continued to meet the remote part of its circumference, the rectangle contained by one of the lines and the part of it without the circle, is equal to the rectangle contained by the other and the part of it without the circle. (*Euc.* III. 36. Cor.)

8. If two chords cut one another in a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other. (*Euc.* III. 35.)

9. In proportionals, the product of the extremes is equal to the product of the means. (*Euc.* V. Supplement 2. Cor. 1.)

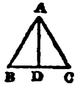


Let ABC be the required triangle, and let the leg AC or BC $=x$. Then (*Euc.* I. 47.) $AB=\sqrt{2x^2}=x\sqrt{2}$; and therefore the perimeter, the sum of all the sides, is $2x+x\sqrt{2}$. Putting this $=p$*, finding x from the equation so obtained, and multiplying the numerator and denominator of its value by $2-\sqrt{2}$, we get $x=\frac{1}{2}p(2-\sqrt{2})$.

For another solution, put $AB=x$. Then (*Euc.* I. 47.) AC or $BC=\sqrt{\frac{1}{2}x^2}=\frac{1}{2}x\sqrt{2}$, and the perimeter will be $x\sqrt{2}+x$. Putting this equal to p, resolving the equation so found for x, and multiplying the numerator and denominator of the result by $\sqrt{2}-1$, we should get $x=p(\sqrt{2}-1)$. The same would be obtained by extracting (§ 169.) the square root of twice the square of the value of x found in the former solution.

By drawing a perpendicular from C to AB, it would be easy to show that the solution, in reference to either of the triangles so formed, would be identical with that of Exam. 7. p. 142.

Exam. 2. Given the difference between the perimeter and perpendicular of an equilateral triangle $=d$; to determine the sides.



Here, let ABC be the triangle, and let AD be drawn perpendicular to BC; then (*Euc.* I. 26.) BD $=$ DC. By putting therefore BD or $DC=x$, the perimeter will be $6x$, and (*Euc.* I. 47.) $AD=\sqrt{(AB^2-BD^2)}=\sqrt{(4x^2-x^2)}=\sqrt{3x^2}=x\sqrt{3}$.

Hence $6x-x\sqrt{3}=d$; whence $x=\dfrac{d}{6-\sqrt{3}}$; or (§ 108.) $\frac{1}{33}(6+\sqrt{3})d$, the double of which is the side.

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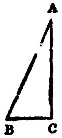
* If we were here to transpose $2x$, square the members so obtained, and resolve the result as a quadratic, we should get $x=\frac{1}{2}p(2\pm\sqrt{2})$, the same as the result in the text, if the double sign were used before $\sqrt{2}$. By using the lower sign in this we get the solution in the text, which is the solution to the question in its literal meaning. By taking the upper sign we get $x=\frac{1}{2}p(2+\sqrt{2})$, which is the leg of the triangle in the question in which p is the *excess* of the sum of the legs above the hypotenuse, as would appear by changing the sign of $\sqrt{2}$ in the equation $2x+x\sqrt{2}=p$.

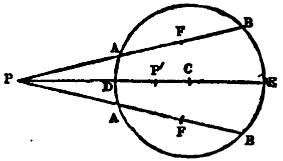
In like manner, in Exam. 2., by transposing $x\sqrt{3}$ and d, by squaring, &c., and resolving the resulting equation, we should get $\frac{1}{33}(6\pm\sqrt{3})d$, a result which would also be obtained by taking AD equal to $\pm\sqrt{3}$: and, by using the lower sign, the expression would be the value of x, if d were the *sum* of the perimeter and perpendicular.

Exam. 3. Given one leg of a right-angled triangle $= 15 \,(=a)$, and the sum of the hypotenuse and twice the other leg $= 33$ $(=b)$; to find the three sides.



Here, if $BC = a$, and $AC = x$, we have (*Euc.* I. 47.) $AB = \surd(a^2+x^2)$; and therefore $2x+\surd(a^2+x^2)=b$. Hence, by transposition, by squaring, &c., we get $x = \frac{1}{3}\{2b \pm \surd(b^2+3a^2)\}$: and, taking $a=15$ and $b=33$, we find $x=8$ and $x=36$, the first of which gives $AB=17$, while the second gives it equal to 39. Of these results 8 and 17 answer the question in its literal meaning, since $2\times 8+17=33$. The others are the answers to the question in which the *excess* of twice the required leg above the hypotenuse is 33, since $2\times 36-39=33$. This will be understood from the circumstance, that the primary equation might have been $2x-\surd(a^2+x^2)=b$, as well as $2x+\surd(a^2+x^2)=b$.

Exam. 4. Through a given point P, to draw a straight line PAB cutting a given circle ABBA, so that the part of it, AB, within the circle may be equal to a given straight line.



To solve this, draw PDE passing through the centre C, and let $PE = a$, $PD = b$, $AB = c$, and $PA = x$. Then $PB = x+c$; and (*Euc.* III. 36.) $PA \,.\, PB = PE \,.\, PD$, or $x(x+c)=ab$. Hence (§ 151.) we get $x = \frac{1}{2}\{-c \pm \surd(c^2+4ab)\}$.*

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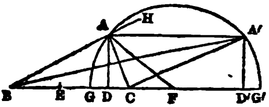
* The positive value of x is PA, agreeably to the assumption; while the negative value, taken positive, is PB. This would appear from taking $PB = x$, and consequently $PA = x-c$, as the equation would then become $x(x-c)=ab$, which would have the same roots as those found above, but with contrary signs. If the given point were within the circle, as at P′, $b\,(=P'D)$ would be negative, and the values of x would become $\frac{1}{2}\{-c \pm \surd(c^2-4ab)\}$. This becomes imaginary, and the problem impossible, if c^2 be less than $4ab$, or, which is the same, if c be less than a chord through P′ perpendicular to DE. Lastly, if P be on the circumference, $4ab$ vanishes, and the value of x becomes 0 or $-c$. The negative values may be explained by supposing the line PB to revolve about the point P, till, not that line itself, but its continuation in the opposite direction, cuts the circle, so as to make the part of it within the circle equal to c. (See *Trigonometry*, Section I.)

A very simple solution would be obtained by bisecting AB in F, and putting $PF = x$. In that case we should have $PA = x-\frac{1}{2}c$, and $PB = x+\frac{1}{2}c$, and therefore $x^2-\frac{1}{4}c^2=ab$; whence $x = \pm\surd(\frac{1}{4}c^2+ab)$. Then the po-

Exam. 5. Given the base of a triangle, its perpendicular, and the ratio of its other sides, to determine the triangle.



Suppose ABC to be the required triangle, and let its base, BC, be bisected in E. Let BE or EC $= a$, the perpendicular AD $= p$, and the ratio of AB to AC that of n to 1; and let ED $= x$. Then BD $= a+x$ and DC $= a-x$; and (*Euc.* I. 47.) we get $AB^2 = (a+x)^2+p^2$, and $AC^2 = (a-x)^2+p^2$. Now, by the question, BA : AC :: n : 1, whence (Note, p. 257.) BA $= n$AC; and, by squaring, $BA^2 = n^2AC^2$, or $(a+x)^2+p^2 = n^2(a-x)^2+n^2p^2$. By performing the operations here indicated, and by transposition, we get $(n^2-1)x^2-2(n^2+1)ax = -(n^2-1)(a^2+p^2)$: and hence, by § 152., and by some easy reductions, we get

$$x = \frac{(n^2+1)a \pm \sqrt{\{4n^2a^2-(n^2-1)^2p^2\}}}{n^2-1}.*$$

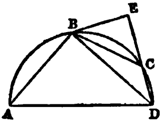
sition of PB would be determined by describing a circle on PC as diameter, and inscribing in it chords each equal to x, and terminated at P.

* Here we have two roots, the less of which is ED, and the greater ED'; and either of the triangles, ABC and A'BC, will answer the conditions of the question. Half the sum of these roots is $\frac{(n^2+1)a}{n^2-1}$, which is evidently EF, F being the point of bisection of DD': and this being independent of p, it is plain that the position of F will be the same for all values of p, so long as a and n are unchanged. By taking, again, half the difference of the two roots, we get DF or D'F $= \frac{\sqrt{\{4n^2a^2-(n^2-1)^2p^2\}}}{n^2-1}$; and, by adding together the square of this and p^2, and by taking the square root, we get (*Euc.* I. 47.) AF or A'F $= \frac{2na}{n^2-1}$, a quantity also independent of p. This remarkable result, in connexion with the value found above for EF, shows that, as long as the base and the ratio of the sides continue the same, the vertex of the triangle will be somewhere on the circumference of the circle whose centre is F, and whose radius is the value found above for AF. That circle, in the language of the ancient geometers, is the *locus* of the vertex. If the perpendicular continually diminish, BA, CA will tend to coincide with BG, CG, and BA', CA' with BG', CG'; and therefore the ratio of BG and CG, and also that of BG' and CG', will be that of n to 1. Hence (*Euc.* VI. 3. and VI. A.), if AG and AG' were joined, the former of the lines thus drawn would bisect the angle BAC, and the latter the exterior angle CAH; and similar conclusions would be obtained by

Exam. 6. To divide a given straight line AB $(=a)$ into two parts, AC and CB, such, that the square of one of them, AC, may be to the rectangle contained by the whole line, and the other part CB, in a given ratio (that of 1 to n).

Let $AC = x$, then $CB = a - x$; and, by the question, $x^2 : a^2 - ax :: 1 : n$. Hence, by equalling the products of the extremes and means, transposing, and resolving the resulting quadratic equation, we get $x = \frac{a}{2n}\{-1 \pm \sqrt{(1+4n)}\}$.*

Exam. 7. Given the chords of three arcs which make up a semicircle; to find its diameter.



To solve this, let $AB = a$, $BC = b$, $CD = c$, and $AD = x$: draw also BE perpendicular to DC produced. Then (*Euc.* II. 12.) $BD^2 = BC^2 + CD^2 + 2CD.CE$. This is easily expressed in terms of a, b, c, and x; since (*Euc.* I. 31.), ABD being a right angle,

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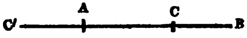
joining A′G and A′G′. It may be remarked in the last place, that, if the perpendicular AD be less than the radius AF, there will be two solutions; that, if it be equal to the perpendicular, there will be only one, the points A and A′ coinciding; and that, if the perpendicular be greater than AF, the problem will be impossible. In connexion with the present investigation, see *Euc.* VI. G.

* This question affords a good instance of the nature and interpretation of negative results. It shows, as is often done by algebraic solutions, that the enunciation of the question is not sufficiently general, but that it ought to be to the following effect: In a given straight line AB, or in its continuation, to find a point such, that the square of the distance of that point from the point A may have a given ratio, that of 1 to n, to the rectangle of the given line and the distance of the required point from B. Now, in the solution given above, we supposed AC to extend from A towards B, and therefore the positive value of x is AC; while, there being a negative root, the line corresponding to it must be AC′, a line lying in the direction opposite to that which was contemplated in the solution. If in the solution we had supposed x to denote AC′, we should have obtained the same values for x, but with contrary signs.



As a numerical example, let $AB = 36$ and $n = 12$. Then $x = 9$, and $x = -12$; that is, $AC = 9$, and $AC' = 12$, each of which will be found to answer.

If $n = 1$, the solution found above will give the section of a line in extreme and mean ratio (*Euc.* II. 11. and VI. 30.): and we thus see we have (*Euc.* I. 47.) $BD^2 = AD^2 - AB^2 = x^2 - a^2$; and, the triangles ABD, ECB being (*Euc.* III. 22. and I. 13.) equiangular; we have (*Euc.* VI. 4.) AD : AB :: BC : CE, or $x : a :: b : CE = abx^{-1}$. Hence $x^2 - a^2 = b^2 + c^2 + 2abcx^{-1}$, or, by multiplying by x, and by transposition, $x^3 - (a^2 + b^2 + c^2)x - 2abc = 0$; an equation, the resolution of which will give the value of the diameter AD.*

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that both Euclid's enunciation and solution of the problem should be more general than they are; as not only one point may be found in the given line which will answer the conditions of the problem as it is proposed by Euclid, but also another in its continuation which will answer to its conditions when it is proposed, as it ought to be, in the extended manner pointed out above.

* This equation will have one positive root and two negative ones, as long as a, b, and c are positive. The greater of the negative roots, taken positive, will be the diameter of a circle such, that if arcs of it be cut off by the given chords, the *difference* between one of them and the other two will be a semicircle. In this case, if b be less than a, the point C will fall between A and B; but, if b be greater than BD, C will lie below AD on the arc of the other semicircle. In solving this, the thirteenth proposition of the second book of *Euclid* is to be employed instead of the twelfth; or, in the solution in the text, b in the one case is to be taken negative, and c in the other. The second negative root will, of course, satisfy the equation; but it is inadmissible as a solution to the question; since, as it is less than some or all of the given chords, they cannot be inscribed in a circle of which it is the diameter. It is plain that the chords may be taken in any order round the circle, as they will always cut off arcs of the same magnitude. Such variations, however, will present no difficulty. It may be remarked, also, that the smaller of the arcs into which the chords divide the circumference are employed throughout.

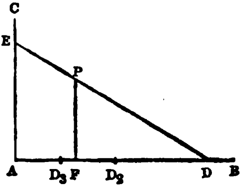
Several modes of solving this question are given by Newton in his *Universal Arithmetic*; and illustrations will be found in Castillioneus's Latin commentaries in his edition of the same work, and also in Wilder's English edition. It is worth remarking, that if the question be solved by means of the known principle (*Euc.* VI. E.), that the sum of the rectangles of the opposite sides of a quadrilateral inscribed in a circle is equal to the rectangle of its diagonals, a root of the final equation will be zero, which is, of course, to be rejected.

It may be proper to remark, that Newton, in connexion with the solutions of this question contained in the fine old work above referred to, has given some directions which are useful, not only in the solution of geometrical problems by means of algebra, but also in solving them geometrically. Of these the following is the substance. When, as is often the case, the diagrams require farther construction, this may be effected by producing some lines till they meet others, or till they become of an assigned length; by drawing from any remarkable point a line parallel or perpendicular to another; by joining remarkable points; by dividing

As a numerical example, let $a = 15$, $b = 15$, and $c = 7$. Then the equation just found becomes $x^3 - 499x - 3150 = 0$; the roots of which are easily found (§ 189.) to be 25, -7, and -18; and the first of these answers the question in its literal meaning.

If, again, $a = 25$, $b = 33$, and $c = 39$, we find x to be 65, $\frac{1}{2}(-65 + \sqrt{265})$, or $\frac{1}{2}(-65 - \sqrt{265})$; the first of which answers.

Exam. 8. Through a given point P between AB and AC, two straight lines which form a right angle, to draw a straight line DE of a given length a.



Since the position of P is given, if we draw PF perpendicular to AB, AF and FP must be known; let therefore the first be denoted by b and the second by c, and let $AD = x$. Then, in the right-angled triangle DAE we have (*Euc.* I. 47.) $AE = \sqrt{(DE^2 - AD^2)} = \sqrt{(a^2 - x^2)}$; and (*Euc.* VI. 4.) the similar triangles DFP and DAE give the analogy, $DF : FP :: DA : AE$; that is $x - b : c :: x : \sqrt{(a^2 - x^2)}$. Hence, by equalling the products of the extremes and the means, squaring, and transposing, we get $x^4 - 2bx^3 + (b^2 + c^2 - a^2)x^2 + 2a^2bx - a^2b^2 = 0$.*

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an oblique-angled triangle into right-angled ones by a perpendicular from one of the angles to the opposite side; or by resolving trapeziums or polygons into triangles by means of diagonals: and in such cases, the formation of similar triangles ought in general to be effected as much as circumstances will permit.

* If the four roots of this equation be all real, one of the positive ones will be AD, and DE will be the required line. Another positive one will give a line such as AD_2, and the required line will be the one drawn through D_2 and P, and continued till it meets AC. The third positive root will give a line such as AD_3, and, if PD_3 be drawn, its continuation from D_3 till it meets CA produced will be the line required. Lastly, the negative root will correspond to a point D_4 in BA produced through A; and, if D_4P be drawn, the part of it between D_4 and AC will be the remaining one of the required lines. If two of the roots be imaginary, a must have been so great in comparison of b and c, that no line equal to it, and passing through P, could lie between the lines forming the angle BAC. If the two greatest of the positive roots be equal, the points D and D_2 will coincide, and there will be but one line in the angle BAC which will answer the question.

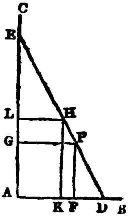
If $c = b$, it is obvious that the two lines, each equal to a in the angle

Exam. 9. **Given the slant height of a cone $= a$ ($= 5$ feet), and its content equal to πb ($= 37{\cdot}6991118$ cubic feet $= 12\pi$); to find its altitude, and the radius of its base.**

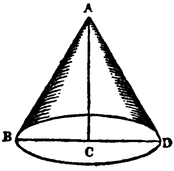
BAC, if there be two, ought to be similarly situated, the one in reference to AB and AC, and the other to AC and AB; and that the two remaining lines should occupy similar situations in the angles in which they lie: and hence we might expect the solution to be simplified; yet, when we take $c = b$ in the equation found above, we get

$$x^4 - 2bx^3 + (2b^2 - a^2)x^2 + 2a^2bx - a^2b^2 = 0,$$

an equation which presents scarcely any more facilities for its resolution than the general one. If, however, as is shown by Lacroix, in his *Application de l'Algèbre à la Géométrie*, x be changed into $by + b$, the equation will become $b^4y^4 + 2b^4y^3 + (2b^4 - a^2b^2)y^2 + 2b^4y + b^4 = 0$; and, by dividing this by b^4, we obtain a reciprocal equation, which (§ 197.) may be resolved as a quadratic. The same object is attained in a very simple and instructive manner in a solution to the following effect, given for this case by Newton in his *Universal Arithmetic*. Bisect DE in H, and draw HK and HL parallel to AC and AB. Then, putting EH or HD $= a$, PF or PG $= b$, and PH $= x$, we have DP $= a - x$ and EP $= a + x$; and, from the equiangular triangles DAE, DFP, and PGE, we get (*Euc.* VI. 4.) DP : PF :: DE : EA, and PE : PG :: DE : DA; that is, $a - x : b :: 2a :$ EA, and $a + x : b :: 2a :$ DA. These two analogies give values of AD and AE; and by putting the sum of the squares of these equal to $4a^2$, the square of DE, and reducing the equation, we get $x^4 - 2(a^2 + b^2)x^2 = 2a^2b^2 - a^4$; whence $x = \pm\sqrt{\{a^2 + b^2 \pm b\sqrt{(4a^2 + b^2)}\}}$.



In connexion with this solution, Lacroix, after remarking that it was given by Newton for the purpose of showing how a judicious selection of the unknown quantity may simplify the solution of a problem, observes that, in determining on the choice of the unknown quantity, we ought to assume that one which, in the circumstances of the question, will undergo the fewest changes. After the solution also, Newton gives a useful rule of which the following is the substance. When, as it is generally easy to see beforehand, two unknown quantities bear such relations to the other quantities concerned in a question, that an equation in every respect similar would be produced whichever of the two might be employed; or when, if both be employed at once, they will have the same dimensions and exactly the same form in the final equation, except perhaps a difference of signs; it is then better to employ neither, but to select some third quantity which bears the same relation to each of them, such as their half sum, their half difference, a mean proportional between them, or the like. Newton's solution of this problem exemplifies both these principles. Thus when AF and FP are equal, PH can evidently have only two values different in magnitude, one for the lines in the angle BAC, and the other for those in the other angles, while AD may have four values; and therefore, according to Lacroix's principle, Here, if we put AC $=x$, we have (*Euc.* I. 47.) $CD^2=a^2-x^2$; and hence (Note, p. 257.) the content of the cone is $\frac{1}{3}\pi x(a^2-x^2)$. Putting this equal to πb, we get, by easy reductions, $x^3-a^2x+3b=0$, an equation which, by Descartes's rule (p. 172.), will have one negative root, and either two positive or two imaginary ones. With the numbers given in the question, it becomes $x^3-25x+36=0$, one value of which is readily found, by § 189., to be 4, and thence, by § 174., the other roots are easily found to be $-2+\sqrt{13}$ and $-2-\sqrt{13}$. Of these three roots, each of the first two answers the question in its literal meaning; but the third, though it satisfies the equation, must be rejected as a solution to the question, as it gives an imaginary value for CD.



Exercises.

1. Given the sum of the perimeter and diagonal of a square $=s$; to find its side.

Ans. $\frac{1}{14}s(4-\sqrt{2})$. If s were the difference of the perimeter and diagonal, the side would be $\frac{1}{14}s(4+\sqrt{2})$.

2. Find the dimensions of a rectangular garden containing an acre, and having its length and breadth in the ratio of 8 to 5.

Ans. Length and breadth 16 perches and 10 perches.

3. Given the area of an equilateral triangle $=a$; to find its side. *Ans.* $\frac{2}{3}\sqrt{3a\sqrt{3}}$.

4. Given the diagonal of a rectangle $=100$ $(=d)$, and its perimeter $=248$ $(=p)$; to find its sides.

Ans. 96 and 28: general values, $\frac{1}{4}\{p\pm\sqrt{(8d^2-p^2)}\}$.

5. Given the two lines drawn from the oblique angles of a right-angled triangle to the points of bisection of the opposite sides, equal to a and b; to determine the sides.

Ans. The legs are $2\sqrt{\frac{4a^2-b^2}{15}}$ and $2\sqrt{\frac{4b^2-a^2}{15}}$, and the hypotenuse $=\frac{2}{5}\sqrt{(5a^2+5b^2)}$.

x should be put to denote PH in preference to AD. According to Newton's rule, again, in this case of the problem, the two lines AD and AE are evidently alike related to the data of the problem, and consequently neither of them should be denoted by x. This, therefore, according to his principle, he employs to denote PH, half the difference of the segments into which the hypotenuse of the triangle DAE is divided

6. Given the hypotenuse of a right-angled triangle $= h$, and the sum of the legs $= s$; to find the legs.

Ans. $\frac{1}{2}\{s \pm \sqrt{(2h^2 - s^2)}\}$, the upper sign giving one leg, and the lower the other.*

7. If the perimeter of a right-angled triangle be p, and the radius of its inscribed circle r, what are its legs? †

Ans. $\frac{1}{2}\{\frac{1}{2}p + r \pm \sqrt{(\frac{1}{4}p^2 - 3pr + r^2)}\}$.

8. Given the base of a triangle $= 2b$, its perpendicular $= p$, and the difference of its other sides $= d$; to determine its sides.

Ans. The difference of the segments into which the base is divided by the perpendicular is equal to $d\sqrt{\left(1 + \frac{4p^2}{4b^2 - d^2}\right)}$.

9. Given the base of a triangle $= 14 = 2b$, the perpendicular $= 12 = p$, and the sum of the sides $= 28 = s$; to find the sides.

Ans. The difference of the segments of the base made by the perpendicular is 4, or, in general terms, $s\sqrt{\left(1 - \frac{4p^2}{s^2 - 4b^2}\right)}$.

10. Given the base of a triangle $= 14 = 2b$, the perpendicular $= 12 = p$, and the sum of the squares of the sides $= 394 = s$; to find the sides.

Ans. Half the difference of the segments into which the base is divided by the perpendicular is 2, or, in general terms, $\sqrt{(\frac{1}{2}s - b^2 - p^2)}$; whence the sides are easily found.

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in P, that triangle having AD and AE as legs, and its hypotenuse being given.

* The following is an outline of perhaps the neatest and simplest mode of solving this question. Let the legs be x and y: then $x + y = s$ and $x^2 + y^2 = h^2$. Take the latter of these from the square of the former, and there will remain $2xy = s^2 - h^2$. Take this again from $x^2 + y^2 = h^2$; then, by extraction, $x - y = \sqrt{(2h^2 - s^2)}$, and the rest is easy. If the hypotenuse and the difference of the legs were given, the solution might be effected in a similar manner.

† Since (*Euc.* IV. 4. cor. 5.) the sum of the legs exceeds the hypotenuse by $2r$, we shall readily find that the hypotenuse is equal to $\frac{1}{2}(p - 2r) = \frac{1}{2}p - r$, and the sum of the legs to $\frac{1}{2}(p + 2r) = \frac{1}{2}p + r$. Hence the solution is obtained from that of the last question, by changing h into $\frac{1}{2}p - r$ and s into $\frac{1}{2}p + r$. A clumsy and lengthened solution of this question is given in the second volume of Bonnycastle's larger *Algebra*, p. 323.

11. Find the sides of a right-angled triangle having its area $= a$, and the sides in arithmetical progression.*

Ans. $\frac{1}{2}\sqrt{6a}$, $\frac{2}{3}\sqrt{6a}$, and $\frac{5}{6}\sqrt{6a}$.

12. Find the sides of a right-angled triangle having its area $= a$, and the sides in geometrical progression.

Ans. The leg, which is a mean between the hypotenuse and the other, is $a^{\frac{1}{2}}(\sqrt{20}+2)^{\frac{1}{4}}$.

13. Prove that when the sides of a right-angled triangle are in geometrical progression, the hypotenuse is divided in extreme and mean ratio by the perpendicular to it from the right angle, and that the greater segment of the hypotenuse is equal to the less leg.

14. Given the sum of the legs of a right-angled triangle $=391 = s$, and the difference of the squares of the hypotenuse and a perpendicular to it from the right angle $= 69{,}121 = d$; to find the hypotenuse and perpendicular.

Ans. 289 and 120: general value of the hypotenuse,

$$\sqrt{\{-\tfrac{1}{3}(s^2-2d)+\tfrac{2}{3}\sqrt{(d^2-s^2d+s^4)}\}}.$$

15. Given the difference of the legs of a right-angled triangle $= 5$, and the perpendicular from the right angle to the hypotenuse $= 12$; to find the legs. *Ans.* 15 and 20.

16. Required the dimensions of a rectangle such, that if its length be increased by a, and its breadth by b, its area will be increased by c; and that if its length be diminished by a_2, and its breadth by b_2, its area will be diminished by c_2.

Ans. The length and breadth are $\dfrac{a_2c-ac_2-aa_2(b+b_2)}{a_2b-ab_2}$, and $\dfrac{b_2c-bc_2-bb_2(a+a_2)}{ab_2-a_2b}$.†

17. Given the content of a right cone $= c\,(=3141{\cdot}593)$, and

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* By assuming the sides equal to $x-y$, x, and $x+y$, and putting the sum of the squares of the first two = the square of the last, we readily find that the sides will be $3y$, $4y$, and $5y$; and therefore it appears that when the sides of a right-angled triangle are in arithmetical progression, they are always as the numbers 3, 4, and 5. This would appear also from the answer found above, $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$ being as 3, 4, and 5.

† The data of this question will sometimes be insufficient for its solution, and may therefore render it indeterminate or absurd. Thus, if a, a_2, b, and b_2, be each equal to the same quantity, d, the question will be indeterminate, if $c-c_2=2d^2$; and, unless this be so, it will in that case be impossible. Modifications will arise also, both in the enunciation of the question and in the results, from the signs of the data. Compare this exercise with Exer. 46. p. 134.

the sum of its altitude and the radius of its base $= s\ (=40)$; to find its dimensions.

Ans. To find the radius of the base, resolve the equation, $x^3 - sx^2 + b = 0$, where $b = 3c\pi^{-1}$. With the numbers given above, this equation becomes $x^3 - 40x^2 + 3000 = 0$, and the roots are 10, $15 + 5\sqrt{21}$, and $15 - 5\sqrt{21}$; the first and second of which answer the question as proposed. The third with its sign changed answers the question in which s is the excess of the radius above the altitude.

18. Given the radius of a sphere $= a$; to find the dimensions of a right cone inscribed in it, and having its solid content $= c$.

Ans. If x be the altitude, its value will be found by resolving the equation, $x^3 - 2ax^2 + b = 0$, where $b = 3c\pi^{-1}$.*

CHAPTER XVII.

ELIMINATION, COMPOUND INTEREST, ETC.†

259. Different methods of elimination in the simplest and most elementary cases have been already given in Chapter VIII. When, however, the proposed equations are of a more complicated kind, such as when products or powers of the unknown quantities are concerned, these methods often fail altogether; and, even when they may be employed, they generally lead to difficult and laborious solutions. The following examples will make the learner

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[*] This equation will have one negative root, which must evidently be rejected as a solution to the question. The other roots, with certain data, will be imaginary, and the problem will then be impossible, the given content being greater than that of the greatest cone that can be inscribed in the given sphere. If they be real and unequal, they will be both positive, and there will be two distinct and dissimilar cones that will answer the question; but if they be equal, there will be but one cone, and that will be the maximum one. As an example, let $a = 25$ and $c = 7542{\cdot}9648$, and consequently $b = 2401$; then the values of x will be 49, $\frac{1}{2}(1 + \sqrt{50})$, and $\frac{1}{2}(1 - \sqrt{50})$, the first and second of which answer.

[†] This Chapter consists of several independent parts, which, either in their own nature, or from the brief manner in which they are here touched on, are too short to form separate chapters.

acquainted, in some degree, with the methods that may be employed in such cases.

Exam. 1. Given $2x^2-3xy+3y^2-5=0$, and $3x^2+xy-2y^2-12=0$; to find x and y.

We might here, by § 151., find a value of x or y from one of the equations, and substitute it, according to § 144., in the other equation: or we might find an expression for one of the unknown quantities from each of the given equations; and then, according to § 143., put the values so found equal to each other. In both cases, however, the reduction of the equations so found would be troublesome on account of the radicals which they would contain.

$$2x^2-3xy+3y^2-5=0 \ldots\ldots (1.)$$

$$3x^2+xy-2y^2-12=0 \ldots\ldots (2.)$$

$$6x^2-9xy+9y^2-15=0 \ldots\ldots (3.)$$

$$6x^2+2xy-4y^2-24=0 \ldots\ldots (4.)$$

$$11xy-13y^2-9=0 \ldots\ldots (5.)$$

$$x=\frac{13y^2+9}{11y} \ldots\ldots (6.)$$

$$2\left(\frac{13y^2+9}{11y}\right)^2-\frac{3(13y^2+9)}{11}+3y^2-5=0 \ldots (7.)$$

$$136y^4-217y^2+81=0 \ldots\ldots (8.)$$

$$y=\pm 1, \text{ and } y=\pm\tfrac{9}{58}\sqrt{29} \ldots\ldots (9.)$$

$$x=\pm 2, \text{ and } x=\pm\tfrac{23}{58}\sqrt{29} \ldots\ldots (10.)$$

The solution, of which what is given in the margin is an outline, is considerably easier than either. In it equations (3.) and (4.) are got from (1.) and (2.) by trebling the first and doubling the second. Then, by subtracting, as in § 140. (3.) from (4.) we obtain (5.), thus not completely eliminating x, but destroying the terms containing its highest power: and by resolving (5.) for x, we get (6.). Equation 7. is found by substituting the value of x in (1.); and (8.) is derived from (7.) by performing the actual operations, clearing the result of fractions, &c. We find (9.) from (8.) by means of § 154.: and, lastly, (10.) is found by substituting the values of y in (6.). It appears, therefore, that x and y may be 2 and 1, or $\frac{23}{58}\sqrt{29}$ and $\frac{9}{58}\sqrt{29}$; or they may be the same with the opposite signs.

Another mode of solving such problems may be obtained by arranging the terms of the given equations according to the powers of one of the quantities, if they be not so arranged already, and then applying to the results the process (§ 81. or 82.) for finding the greatest common measure, till a remainder shall be found which will contain only one of the unknown quantities. Then, by putting this remainder equal to nothing, an equation will be obtained, the roots of which will be the values of that quantity. Thus, in the present example, by multiplying the second equation by 2, and dividing the result by the first, we get 3 as quotient, with the remainder $11xy-13y^2-9$. Then, for avoiding fractions, we multiply the first equation by $11y$; and dividing the product by the foregoing remainder, we get as quotient $2x$, and as remainder $-7xy^2+18x+33y^3-55y$. Multiplying this remainder by 11, and continuing the division, we obtain as quotient $7y-18$, and as remainder $-272y^4+434y^2-162$. Lastly, putting this remainder equal to 0, and dividing by -2, we get $136y^4-217y^2+81=0$, as before.

As to the principle on which this method depends, it is plain, that if there be two expressions each equal to nothing, and if one of them be divided by the other, the remainder, whatever may be its form, must also be equal to nothing. Thus, if in dividing M by N, we get Q as quotient and R as remainder, we have, by multiplication, $M=NQ+R$, which for $M=0$, and $N=0$, gives also $R=0$. Hence, therefore, in every such process as the one indicated above, each remainder must be equal to nothing; and it thus appears that we are entitled to assume in the present instance, $-272y^4+434y^2-162=0$, and that a like assumption may be made in all similar cases.

Exam. 2. As an example of a more general kind, let the equations, $Y_3x^3+Y_2x^2+Y_1x+Y_0=0$, and ${}_3Yx^3+{}_2Yx^2+{}_1Yx+{}_0Y=0$, be proposed, in which Y_3, ${}_3Y$, &c., are expressions formed by the combination of y and given numbers, and let it be required to eliminate x.

To effect this, multiply the first by ${}_3Y$ and the second by Y_3; then, by taking the difference of the products, and by putting for brevity, ${}_3Y\,Y_2-Y_3\,{}_2Y=\Upsilon_2$, ${}_3Y\,Y_1-Y_3\,{}_1Y=\Upsilon_1$, and ${}_3Y\,Y_0-Y_3\,{}_0Y=\Upsilon_0$, we obtain $\Upsilon_2x^2+\Upsilon_1x+\Upsilon_0=0$. Again, by multiplying the first of the given equations by ${}_0Y$ and the second by Y_0, by taking the difference of the products, by dividing the remainder by x, and by putting ${}_0Y\,Y_3-Y_0\,{}_3Y={}_2\Upsilon$, ${}_0Y\,Y_2-Y_0\,{}_2Y={}_1\Upsilon$, and ${}_0Y\,Y_1-Y_0\,{}_1Y={}_0\Upsilon$, we get at length ${}_2\Upsilon x^2+{}_1\Upsilon x+{}_0\Upsilon=0$. By multiplying this result by Υ_2 and the former one by ${}_2\Upsilon$, and taking the difference of the results; and also by multiplying the former by ${}_0\Upsilon$ and the latter by Υ_0, and by taking the difference of the products, and dividing it by x; two results will be obtained, each involving only the first power of x. Eliminating x, therefore, between these, we should obtain an equation containing only known functions of y; and by substituting for these their values, we should get an equation, the resolution of which would give the values of y.*

Exam. 3. Required the values of x and y from the equations, $8x^3-2x^2y+3xy^2-27y^3=0$, and $16x^4-32x^3y+27y^4=0$.

To solve this and similar questions in which in each of the equations the sums of the indices of x and y in the several terms are equal, we may assume $y=xz$. Then, by substituting this in the two equations, and dividing in the first by x^3 and in the second by x^4, we get $8-2z+3z^2-27z^3=0$, and $16-32z+27z^4=0$. If these equations have one or more values of z in common, any such value will furnish a solution of the question; but if there be no value in common, the original equations cannot co-exist, and the question is impossible. We might, therefore, find the values of z in the foregoing equations, and select any that might agree. An easier mode, however, is to find by § 81. or § 82., the greatest common divisor of the first members of those equations, and to put it equal to nothing. We thus get $2-3z=0$, and consequently $z=\frac{2}{3}$. Hence, by the original assumption, we have $y=\frac{2}{3}x$. The question, therefore, is unlimited, as x may be taken at pleasure, and y will be two thirds of it.

Exercises. †

1. Find the values of x and y in the equations, $x^3-3yx^2+(3y^2-y+1)x-y^3+y^2-2y=0$, and $x^2-2yx+y^2-y=0$.

Ans. $x=0$ and $y=0$; or, $x=2$ and $y=1$.

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* This method labours under the disadvantage of generally giving the final equation for determining y of an unnecessarily high order. The solution might also be obtained by finding the greatest common measure as in the first example, and by that means, with proper management, the degree of the equation may be prevented from rising so high as in the method followed above. In both methods, however, the multiplications that are required frequently introduce roots into the final equations which are not admissible as solutions, and which must therefore be rejected. For further information on this subject, see Fourcy, *Algèbre*, chap xvii. See also Euler, Bezout, &c.

† These exercises occur in the *Algebras* of Fourcy, Reynaud, and others.

2. Resolve the equations, $yx^2+9x-10y=0$, and $(y-1)x^2+2x-5y+3=0$.

Ans. $x=1$ and $y=1$; $x=2$ and $y=3$; $x=-5-\sqrt{10}$ and $y=\frac{3}{5}(-1+\sqrt{10})$; or $x=-5+\sqrt{10}$ and $y=\frac{3}{5}(-1-\sqrt{10})$.

3. Resolve the equations, $x^2+(8y-13)x+y^2-7y+12=0$, and $x^2-(4y+1)x+y^2+5y=0$.

Ans. $x=3$ and $y=1$; $x=2$ and $y=1$; $x=1$ and $y=0$; or $x=1$ and $y=-1$.

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260. As a practically useful subject of inquiry, it may be proper here briefly to investigate the modes of performing computations regarding compound interest and annuities. The person who lends money is paid by the borrower, for the use of it, a certain sum which is called its *interest,* and the money lent is called the *principal.* When the interest is paid as it becomes due, or does not itself bear interest, the money is said to be lent at *simple interest;* but when the interest, instead of being paid, is added to the principal, and bears interest along with it, the interest which thus accumulates is called *compound interest.* The money which is paid for the use of one hundred pounds for the interval between two periodical payments, such as a year, half a year, or the like, is termed the *rate;* and the sum due for both principal and interest is called the *amount.* Now, if p be put to denote the principal, a the amount, and r the rate, the amount of £100 at the end of one of the intervals of time would be $100+r$. The hundredth part of this, which we may denote by a_1, will evidently be the amount of £1 for the same time; and pa_1 will plainly be the amount of the principal p for the same period. Taking this as a new principal and multiplying it by a_1, we shall have its amount at the end of the next period equal to pa_1^2, which is evidently the amount of p at the end of two periods. By pursuing this reasoning, it would appear, that, at the end of n years, or other equal periods, we should have the amount $a=pa_1^n$. Hence it appears, that, *to find the total amount of the original principal and of the interest accumulated during n equal periods, we are to raise the amount of £1 for one of the periods to the nth power, and to multiply the result by the original principal.*

261. Hence, conversely, if the amount be given, to find the original principal, or, as it is usually called in that case, the *present worth,* we have merely to divide the given amount by a_1^n.

Exam. 4. If a father, at the birth of a child, set apart £1000, and allow it to accumulate at compound interest, at the rate of 5 per cent per annum, what will be the amount at the end of twenty-one years?

Here, dividing $100+5$ by 100, we get $a_1=1·05$, the twenty-first power of which is 2·785963 *; and multiplying this by 1000, we get £2785·963, or £2785 19*s*. 3*d*., the amount required.

Exam. 5. What would the amount have been in the last example, had the interest been payable half-yearly, instead of yearly?

In this case we have $a_1=1·025$; which, as there would be forty-two payments, must be raised to the forty-second power. This is found to be 2·820996, the product of which by 1000

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[*] The method of obtaining such powers by the contracted process for the multiplication of decimal fractions will be found in the Author's *Treatise on Arithmetic*, 25th edition, p. 209. Tables also for facilitating computations in compound interest and annuities will be found in the same work, commencing in p. 268.

When the student has become acquainted with the nature and use of logarithms, he will find that those remarkable numbers are employed with great advantage in such investigations as the present. Thus, by taking the logarithms of the quantities in the equation found in § 260., we get $\log a=\log p+n\log a_1$; an expression which, without change, will solve Exam. 4. with great facility. By transposition we get $\log p=\log a-n\log a_1$, which will solve Exam. 6. By another transposition, and by dividing by $\log a_1$, we get $n=\frac{\log a-\log p}{\log a_1}$; an expression which will show in what time a given principal at an assigned rate would yield a given amount, a problem which does not admit of being solved by ordinary algebraic means.

As an example of the use of this latter expression, let it be required to find in what time money lent at compound interest at 4½ per cent per annum would "double itself;" that is, in what time the amount would be double of the original principal. Here $a_1=1·045$, of which, by the tables, the common logarithm is 0·019116. Let also $p=1$, and consequently $a=2$. Then, $\log p=0$, and $\log a=0·301030$. Hence, from the expression found above, by dividing 0·301030 by 0·019116, we get $n=15·7475$, or fifteen years and three quarters nearly. In a similar way we might find in what time the principal would be trebled, or increased in any assigned degree.

From the expression found above we should also get $\log a_1=\frac{\log a-\log p}{n}$; an expression, by means of which we might find the rate at which a given principal would be increased to a given amount in an assigned time.

is £2820·996, or £2820 19*s.* 11*d.*, the amount required. We thus obtain an amount greater by £35 0*s.* 8*d.* than the former one, in consequence of the greater frequency of the payments, as by this means certain portions of the interest are sooner brought themselves to bear interest.

Exam. 6. How much must a person lay out, to accumulate, at 4 per cent per annum, compound interest, so that in ten years the amount may be £1500?

We have here $a_1 = 1·04$, the tenth power of which is found to be 1·480244. Dividing, therefore, £1500 by this, we get £1013·3465, or £1013 6*s.* 11¼*d.*, the answer.

Exam. 7. If the population of a town containing 100,000 inhabitants increase at the average annual rate of 2 per cent, what will be its amount at the end of ten years? *

Here $a_1 = 1·02$; and therefore $a_1^{10} = 1·21899$, the product of which by 100,000 is 121,899, the amount required

Exam. 8. The population of Glasgow was 202,426 in the year 1831, and 282,134 in 1841. At what average rate per cent per annum must the population have increased during the interval between these times?

Here we have $p = 202,426$, $a = 282,134$, and $n = 10$. Then by resolving the equation found in § 260. for a_1, we shall find that, to get this quantity, we have merely to divide a by p †, and to take the nth root of the quotient. By this means we find in the present instance $a_1 = 1·03375$. Then, by multiplying this by 100, and subtracting 100 from the product, we find the required rate r to be 3·375. The reason of this is plain from the § just referred to.

262. An *annuity* is a sum of money which is payable at equal intervals of time, such as salaries, rents, or other sources of income, payable yearly, half-yearly, or quarterly.

One of the principal problems regarding annuities, and the one whose solution leads to those of all others, is that in which it is required to find the amount or entire sum due for an annuity remaining unpaid for a given period, compound interest being allowed on all the money thus remaining unpaid. To investigate the solution of this, let a_1 and n remain as in § 260., and let u be the annuity and m its amount at the end of n of the equal intervals. Then, at the end of the first year or interval, the amount unpaid will be simply u; but, at the end of the second, there will be due another annuity together with ua_1, the former year's annuity with its interest; that is, the amount due will be $u+ua_1$, or $u(1+a_1)$. In like manner, at the end of the third period, the whole amount due would be $u+u(1+a_1)a_1$, or $u(1+a_1+a_1^2)$; and it would appear similarly, that for n periods we should have $m=u(1+a_1+a_1^2+\ldots+a_1^{n-1})$. Now, the series within the vinculum is a geometrical progression, having a_1 for ratio. Finding its sum, therefore, by § 137., we get $m = \dfrac{(a_1^n-1)u}{a_1-1}$.*

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* Questions of this or any similar kind, in which the increase or diminution of a quantity is proportional to the quantity itself, will evidently be solved on the same principle as questions in compound interest.

† This quotient in the present instance is 1·39376; the product of which by 100 is 139·376. Subtracting 100 from this, we have remaining 39·376, the rate per cent of the increase of the population during the whole ten years. The tenth root of 1·39376 will be easily found by resolving the equation $x^{10} = 1·39376$ by Horner's method, and more easily still by means of logarithms.

Exam. 9. If a person could save £100 a year out of a salary payable yearly, and could improve it at the rate of 6 per cent per annum, compound interest, what would be the amount of the savings and interest at the end of thirty years?

Here $a_1 = 1{\cdot}06$, and $a_1^{30} = 5{\cdot}743491$. Taking 1 from each of these, and dividing the second remainder by the first, we get 79·058183, the amount of an annuity of £1 for thirty years at 6 per cent per annum, compound interest; and, multiplying this by 100, we get £7905 16*s.* 4½*d.*, the amount required.

Exam. 10. Suppose every thing to be as in the last example, except that both the salary and the interest are payable half-yearly: what will be the amount?

In solving this, every thing will be as in the last example, except that $a_1 = 1{\cdot}03$; that $a_1^{60} = 5{\cdot}891603$; that the amount of an annuity of £1 is 163·05343; and that the product of this by £50 is £8152 13*s.* 5*d.*, the amount required. This amount is nearly £250 greater than the former, on account of the frequency of the payments.

263. The present worth of an annuity, to continue for a given

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* If $u = 1$, this formula will afford the means of computing a table exhibiting the amount of an annuity of £1 for an assigned number of years, or other intervals, at a given rate per cent. See *Arithmetic*, p. 269.

time, will be obtained (§ 261.) by dividing its amount found by the last §, by a_1^n. Hence, denoting the present worth by w, we shall have $w=\frac{(1-a_1^{-n})u}{a_1-1}$. Comparing this result with what is established in § 261., we see that the present worth of an annuity of £1 is found by taking its present worth at compound interest from unity, and dividing the remainder by a_1-1.

Exam. 11. If a person have an annual profit rent of £75, which is payable yearly, and is to continue 32 years, how much ought he to get for it at present, allowing the purchaser compound interest at 4 per cent per annum on what he pays for it?

Here $a_1=1{\cdot}04$, the 32d power of which is 3·508059. Dividing 1 by this we get 0·285058; the difference between which and 1 is 0·714942; and, by dividing this by 0·04, we obtain 17·87355, which is the present value of £1 of the annuity. Multiplying it therefore by 75, we get £1340 10*s.* 4*d.*, the required price.

Exam. 12. A person who owes £1200, wishes to extinguish the debt by twelve equal annual payments: what must be the amount of each payment, compound interest, payable yearly, being allowed at 5 per cent per annum?

This question is the converse of the last, as we are here to find the annuity of which the present value is £1200. From the equation, therefore, found above, we get $u=\frac{(a_1-1)w}{1-a^{-n}}$; and as $w=$ £1200, $a_1=1{\cdot}05$, and $n=12$, we readily find u, the required annual instalment, to be £135 7*s.* 9¾*d.*

264. An annuity which is to continue for ever is called a *perpetuity*. In reference to such an annuity n is infinite, and (§ 126.) the expression in the last § becomes simply $w=\frac{u}{a_1-1}$.

Exam. 13. If a person have in perpetuity a property worth 500 guineas a year, for how much should it sell, the purchaser being allowed 4½ per cent per annum for his money?

Here $a_1-1=0{\cdot}045$; and by dividing £525 by this, we get £11666 13*s.* 4*d.* the required price.

265. An annuity which does not come to be possessed till the end of some assigned time is said to be in *reversion*. Thus, if one person is to enjoy the benefit of an annuity for n years, and

another for the next r years, it is said to be an annuity in reversion in reference to the latter. The obvious method of finding the present value of the reversion is to subtract the present value of the annuity for n years from its present value for $n+r$ years. Hence, if v be the present value of the reversion, we shall have, by § 263. and by obvious reductions,

$$v=\left(\frac{1-a_1{}^{-n-r}}{r-1}-\frac{1-a_1{}^{-n}}{r-1}\right)u=\left(\frac{a_1{}^{-n}-a_1{}^{-n-r}}{r-1}\right)u$$

$$=\left\{\frac{1-a_1{}^{-r}}{(r-1)a_1{}^{n}}\right\}u=\left\{\frac{a_1{}^{r}-1}{(r-1)a_1{}^{n+r}}\right\}u,$$

any of which expressions will give the required value.

Exam. 14. What is the present value of an annuity of £112 10*s.* 0*d.* to commence at the end of 10 years, and to continue 20 years, at 4 per cent per annum?

Here $n=10$, and $n+r=30$; and by raising 1·04 to the 10th and 30th powers, and employing the first, second, or fourth of the foregoing expressions for v, we get £1032 17*s.* 6½*d.*, the present value of the reversion.

Exam. 15. A person bequeaths to one charity for 10 years a property which produces annually the clear sum of £625; to another he leaves it for the next 17 years; and to a third he bequeaths it for all time coming. What are the present values of the three bequests at 4 per cent per annum, compound interest?

In solving this question we have $a_1=1{\cdot}04$, and we find the first answer, by means of § 263., to be £5069 6*s.* 2¼*d.*, taking $n=10$. Then, by means of § 265., taking $n=10$ and $r=17$, we get £5136 13*s.* 7¼*d.* as the value of the first reversion; and by means of the same §, in connexion with § 264., we find the value of the second reversion to be £5419 0*s.* 2½*d.*[*]

Exam. 16. Required the value of the continual product $\sqrt{a\sqrt{a\sqrt{a\ldots\ldots}}}$, carried out without limit.

To find the value of this, assume it equal to x. Then, by squaring both members, we get $x^2=a\sqrt{a\sqrt{a\ldots}}$, or, as is evident, $x^2=ax$; whence, by dividing by x, we find $x=a$.

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[*] For exercises and other examples, and for various additional details regarding compound interest and annuities, the reader is referred to the articles on those subjects in the Author's *Treatise on Arithmetic*: and other exercises will be found in the same work in the "Miscellaneous Questions," commencing in p. 237. 25th edition.

The same result might also be obtained by expressing the proposed quantity under the form, $a^{\frac{1}{2}}a^{\frac{1}{4}}a^{\frac{1}{8}}a^{\frac{1}{16}},\ldots, a^{\frac{1}{2^n}}$, to which it is evidently equivalent. Now (§ 105.) the product of these factors is a with an index equal to the sum of their indices, that is (§ 137.), $1-\frac{1}{2^n}$; an expression which (§ 126.) becomes 1 when n is infinite. The product, therefore, of an infinite number of the factors is simply a.

Exam. 17. Find the value of $\sqrt{a\sqrt[3]{b\sqrt{a\sqrt[3]{b\ldots}}}}$, continued infinitely.

Putting this equal to x, and first squaring both members, and then cubing the result, we get, evidently, $x^6=a^3bx$, whence $x=a^{\frac{3}{5}}b^{\frac{1}{5}}$.

This may also be put under the form, $a^{\frac{1}{2}}b^{\frac{1}{6}}a^{\frac{1}{12}}b^{\frac{1}{36}}a^{\frac{1}{72}}b^{\frac{1}{216}}\ldots$. Now (§ 138.) the sum of the infinite series $\frac{1}{2}$, $\frac{1}{12}$, $\frac{1}{72}$, &c., the indices of a, is $\frac{3}{5}$, and the sum of the indices of b is evidently one third of this, or $\frac{1}{5}$. Hence, $x=a^{\frac{3}{5}}b^{\frac{1}{5}}$, as before. If $b=a$, this will become $x=a^{\frac{4}{5}}$.

Exercises.

4. Find the value of the continual product of the foregoing kind, in which $\sqrt{a\sqrt[3]{b\sqrt[4]{c}}}$ recurs perpetually. *Ans.* $(a^{12}b^4c)^{\frac{1}{23}}$.

5. Find the value of the continual product of the same kind, produced by the repetition without end of the expression, $\sqrt[3]{a\sqrt{b\sqrt[3]{c^{-1}}}}$. *Ans.* $(a^6b^4c^{-1})^{\frac{1}{17}}$.

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266. When, as is often done, cannon balls of equal size are piled up in the form of a pyramid, the number of balls in the pile may be determined by some of the methods that have been given for finding the sums of series. The base of the pyramid may be either a square or an equilateral triangle; and when the pyramid is made to contain as many balls as possible, it is plain that each stratum or layer of balls will be a square in the one case and an equilateral triangle in the other, and that each side of

the second layer will contain one ball fewer than a side of the first, a side of the third one fewer than a side of the second, and so on; and that the pyramid will terminate in a single ball. Hence, if we take the layers downward, the number of balls in the square pyramid will evidently be $1^2+2^2+3^2+\ldots\ldots+n^2$, n being the number of the layers, and consequently the number of balls in each side of the base. Now (Exam. 8. p. 256.) the sum of this series is $\frac{1}{6}n(n+1)(2n+1)$.

267. If the base of each layer be an equilateral triangle, and if the number of balls in one of its sides be n, the whole number of balls in the layer will be $1+2+3+\ldots\ldots+n$, or (§ 133.) $\frac{1}{2}(n^2+n)$. Taking in this n successively equal to 1, 2, 3, &c., we find the number of the balls contained in the several layers to be $\frac{1}{2}(1^2+1)$, $\frac{1}{2}(2^2+2)$, $\frac{1}{2}(3^2+3)$, ..., $\frac{1}{2}(n^2+n)$ *; the sum of which is $\frac{1}{2}(1^2+2^2+3^2+\ldots\ldots+n^2)+\frac{1}{2}(1+2+3+\ldots\ldots+n)$, or (Exam. 8. p. 256. and § 133.) $\frac{1}{12}n(n+1)(2n+1)+\frac{1}{4}n(n+1)$, or finally, by easy reductions, $\frac{1}{6}n(n+1)(n+2)$.

Exercises.

6. How many balls are there in a pile in the form of a square pyramid of the kind above described, having 36 balls in each side of its base, and in another having 50 in each of its sides?

Ans. 16206 and 42925.

7. How many balls are there in a pile having its base an equilateral triangle, each side of which contains 60 balls, and how many in another having 30 balls in each side?

Ans. 37820 and 4960.

8. How many balls are there in the frustum of a square pyra-

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* The numbers 1, 3, 6, 10, 15, &c., thus obtained are called *triangular numbers*, from their being the numbers of balls, dots, &c., disposed at equal distances, and forming equilateral triangles having severally 1, 2, 3, &c., in their sides. The series 1, 3, 6, 10, &c., is the third order of an interesting class of numbers which have been called *figurate numbers.* The first order of these is the series, 1, 1, 1, &c.; the second, 1, 2, 3, 4, &c.; the fourth, 1, 4, 10, 20, 35, &c.: and, in general, the nth order is $\frac{1.2.3.\ldots\ldots(n-1)}{1.2.3.\ldots\ldots(n-1)}$, $\frac{2.3.4.\ldots\ldots\ldots n}{1.2.3.\ldots\ldots(n-1)}$, $\frac{3.4.5.\ldots\ldots(n+1)}{1.2.3.\ldots\ldots(n-1)}$, &c. See the section on the Summation of Infinite Series in the Author's *Treatise on the Differential and Integral Calculus*, where it is shown that the sum of x terms of the nth order is $\frac{x(x+1)(x+2)\ldots(x+n-1)}{1.2.3.\ldots\ldots n}$. The sums of all such series may also be obtained by the method already pointed out in § 258.

mid having 48 balls in each side of its greater base, and 20 in each side of its less one? *Ans.* 35554.

268. Besides the two kinds of piles mentioned above, there may be oblong ones, that is, piles having their bases rectangles which are not squares. In this case the pile will evidently terminate in a single row of balls, the number of which will be greater by one than the difference of the numbers contained in the greater and less sides of the base. Hence, if r be the difference of these last numbers, $r+1$ will be the balls in the top row, $2(r+2)$ those in the next layer, $3(r+3)$ those in the third, and in general $n(r+n)$, where n will be the number of balls in the less side of the base, and $r+n$ those in the greater. The sum of these is

$$r+2r+3r+\ldots\ldots+nr+1^2+2^2+3^2+\ldots\ldots+n^2,$$

$$\text{or } \tfrac{1}{2}n(n+1)r+\tfrac{1}{6}n(n+1)(2n+1);$$

or which is the same, $\frac{1}{6}n(n+1)(3r+2n+1)$.

Exer. 9. Required the number of balls contained in two piles, in one of which there are 60 balls in the greater side of the base, and 40 in the less; while in the other there are 65 in the one side, and 35 in the other? *Ans.* 38540 and 33810.

Exam. 18. Find how many balls must be in each side of the least square pyramid that will contain 30,000 balls. Determine also the upper base of the least frustum of the same pyramid, that will contain the 30,000 balls.

Here, by putting the expression at the end of § 266. equal to 30,000, by actual multiplication, and by transposition, we get $2n^3+3n^2+n-180{,}000=0$; where we easily find, as in pp. 180, 181, &c., that the real value of n lies between 44 and 45. Now, 44 being too small, and fractional values being excluded by the nature of the question, the side of the base must contain 45 balls. The pyramid on this base, however, will contain (§ 266.) 31,395 balls, exceeding the given number by 1395: and therefore, to find the required frustum, we should have to cut off from the top a pyramid containing 1395 balls. The number of balls in each side of the base of this, found as above, would lie between 15 and 16. The former of these must be employed, as the latter would give too large a pyramid, and would therefore leave too small a frustum. Hence, the frustum would have 45 and 16 balls in the sides of its bases; and it would be readily found to contain 30,155 balls. The 30,000 balls, therefore, would leave the upper layer incomplete, as 155 balls would be wanting, so that instead of 256 ($=16^2$) balls, it would contain only 101.

NOTES.

Note A.

Full Elementary Solution of an easy Question.

The following is the algebraic solution of a simple question with full illustrations. The learner will find it useful to read it with care and attention, before he enters formally on the reading and study of the definitions and elementary principles.

Required a number such, that, if 5 be added to its treble; and if, again, the number itself be subtracted from 13, one eighth of the former result shall be equal to the latter.

In solving this question, to avoid the frequent repetition of the expression, "the required number," which must evidently occur very often, we denote that number by the simple character x. Now, whatever that number may be, we naturally express the trebling of it or the multiplying of it by 3, by writing the number 3 before it, so that the product is expressed by $3x$; just as, in common language, the expression three (or 3) yards denotes the treble of the single object, one yard. We are next, by the question, to add 5 to $3x$. Now, as it may be readily anticipated, that the process of addition will be of very frequent occurrence in the solution of questions, and in other algebraic operations, algebraists have agreed to adopt the simple arbitrary sign $+$, instead of the expression "increased by," or some other synonymous one; and this sign they have called *plus*, the Latin word corresponding to the English one, *more*. In the present case, therefore, we have $3x+5$, which is read simply *three x plus five*, that is, *thrice x more by five*. We are now, by the question, to subtract x from 13: and, as the operation of subtraction may likewise be expected to occur very frequently, it has been agreed to denote it by prefixing to what is to be subtracted the simple arbitrary sign $-$, called *minus* (the Latin word for *less*); so that this sign will be a substitute for the expression, "wanting," "diminished by," or the like. In this way we get $13-x$, which is read *thirteen minus x*, that is, *thirteen less by x*. Now, by the question, if the former result, $3x+5$, be divided by 8, the quotient will be equal to $13-x$: and, as division is another elementary operation that must often occur, it must be of advantage to devise some simple mode of expressing it. The mode that is generally employed is to write the divisor below the dividend with a line between them, in the manner that has been adopted for expressing arithmetical fractions. In the present case, therefore, we have $\frac{3x+5}{8}$, which is read for brevity, *three x plus five by eight*, that is, *divided by eight*. This, we have seen, is to be equal to $13-x$; and, as the relation of equality must be of very frequent occurrence, algebraists have agreed to represent it by means of the character $=$, placed between the equal quantities. This sign is read *equal to*, or simply *equal*, for brevity. Hence, we have, in the present question,

$\frac{3x+5}{8}=13-x$; an expression which has the great advantage of combining all the conditions of the problem in a very condensed form, and by means of a small number of characters. We may regard it, therefore, as *the translation of the conditions of the question into the language of algebra*. This expression, from the equality of the parts before and after the sign $=$, is called an *equation*; and those two parts are called its *members*, or, less appropriately, its *sides*.

Now, as our ultimate object is to find the required number x, it is plain that we must separate it and the other numbers with which it is connected in the equation; or, as it is briefly expressed, we must *resolve the equation*. This will be effected by deriving from the equation above found a succession of others, the separation referred to being always kept in view. First, then, we derive from that equation another of a simpler form by multiplying both its members by the denominator 8. By this means, since, by the nature of multiplication and division, 8 times the quotient obtained by dividing $3x+5$ by 8 is $3x+5$, while 8 times $13-x$ is plainly $104-8x$; and since, if equals be multiplied by the same, the products are equal, we get the new equation, $3x+5=104-8x$. Now, the second member of this being less than 104 by $8x$, if we add $8x$ to that member, we get 104. Hence, that we may still have equality, we add $8x$ to each member; and since, if the same be added to equals, the wholes are equal, we get $3x+8x+5=104$, or $11x+5=104$, by incorporating $3x$ and $8x$. Then, since, if the same be taken from equals, the remainders are equal, if we take 5 from each member of the last equation, we get $11x=104-5$, or $11x=99$. Hence, in the last place, since, if equals be divided by the same, the quotients are equal, if we divide each member by 11, we get $x=9$, the required number.

By bringing together the entire work of resolving the equation, it will stand as in the margin; and by examining the process we shall discover some relations that are deserving of notice. Thus we find that, in deriving equation (2.) from (1.), the *division* by 8 is done away by *multiplying* by that number. In deriving (3.) from (2.), the *subtraction* of $8x$ is done away by the *addition* of the same; while, in finding (4.), the *addition* of 5 in (3.) is neutralised by *subtracting* it: and in the last place, in deriving (5.) from (4.), the *multiplication* by 11 is done away by *dividing* by that number. We thus see that, in the solution of this problem by algebra, the process is *analytical*, as we constantly employ operations which are the contraries of those indicated in the equations; and thus, by retracing our steps, as it were, we re-ascend to the value of the required number: and this will be found to be the case in every application of algebra in the solution of problems. As algebra was originally employed almost exclusively in this way, and as it still continues to be employed most extensively as an instrument of investigation or analysis, the term *analytical* has come to be used by mathematicians as almost synonymous with

$$\begin{array}{ll}
\frac{3x+5}{8}=13-x \ldots\ldots\ldots & (1.) \\
3x+5=104-8x \ldots\ldots & (2.) \\
\left.\begin{array}{l} 3x+8x+5=104, \text{ or} \\ 11x+5=104 \end{array}\right\} \ldots & (3.) \\
\left.\begin{array}{l} 11x=104-5, \text{ or} \\ 11x=99 \end{array}\right\} \ldots\ldots & (4.) \\
x=9 \ldots\ldots\ldots\ldots\ldots & (5.)
\end{array}$$

the term *algebraical*. In strictness, however, the latter has a more extensive meaning than the former, as algebra may be used in the demonstration of theorems, and may therefore be *synthetical* as well as analytical.

From the illustrations given above, we see the propriety of employing signs or symbols to denote quantities, operations, and relations, that are of frequent occurrence. Such symbols, however, should be sparingly and judiciously employed. The introduction of too many renders the subject intricate and forbidding, and tends to produce confusion. Few subjects, indeed, are of more importance in algebra and its applications, than the use of a simple, uniform, and easily recollected notation.*

The examples in Section I. will afford other illustrations in addition to those given in this Note.

NOTE B.

Cardan's Solution of Cubic Equations, and Euler's Solution of Equations of the Fourth Degree.

1. WHILE the method of resolving numerical equations given in Chapter X. is much preferable in practice to any other, it may be proper here to point out briefly two modes of resolving equations of the third and fourth degrees, which give the roots in terms of the coefficients, as is done with regard to equations of the first and second degrees.

First, then, let it be required to resolve $x^3+ax+b=0$†, an equation of the third degree; and the investigation will be as in the margin. In this we assume, in equation (2.), $x=y+z$, and the substitution of this in (1.) gives (3.) by § 56. We then assume $3yz+a=0$, as

$$x^3+ax+b=0 \quad \dots\dots \quad (1.)$$

$$x=y+z \quad \dots\dots \quad (2.)$$

$$y^3+z^3+3yz(y+z)+a(y+z)+b=0 \quad \dots\dots \quad (3.)$$

$$3yz+a=0 \quad \dots\dots \quad (4.)$$

$$y^3+z^3+b=0 \quad \dots\dots \quad (5.)$$

$$yz=-\tfrac{1}{3}a, \text{ and } y^3+z^3=-b \quad \dots\dots \quad (6.)$$

$$y^3-z^3=\sqrt{(b^2+\tfrac{4}{27}a^3)} \quad \dots\dots \quad (7.)$$

$$y^3-z^3=2\sqrt{[(\tfrac{1}{2}b)^2+(\tfrac{1}{3}a)^3]} \quad \dots\dots \quad (8.)$$

$$y=\sqrt[3]{\{-\tfrac{1}{2}b+\sqrt{[(\tfrac{1}{2}b)^2+(\tfrac{1}{3}a)^3]}\}} \quad \dots\dots \quad (9.)$$

$$z=\sqrt[3]{\{-\tfrac{1}{2}b-\sqrt{[(\tfrac{1}{2}b)^2+(\tfrac{1}{3}a)^3]}\}} \quad \dots\dots \quad (10.)$$

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* As far as circumstances in particular cases will conveniently permit, notation ought to be *suggestive* in one way or other, for the purpose of assisting the memory. Thus, as p is the initial letter of the words *product* and *principal*, it may in some investigations be used to denote one of these and in some the other; while, for a like reason, d may represent the difference or the half difference of two quantities. The *order* also in which quantities occur may be conveniently expressed by means of subscribed numbers. Thus, t_1, t_2, and t_n will very naturally and properly denote the first, second, and nth terms of a series; while, for an analogous reason, s_1, s_2, and s_n may be used to express respectively the sum of the first powers of certain quantities, the sum of their second powers, and the sum of their nth powers.

† In both this investigation and the next, the equation is supposed to want the second term. Should this not be the case, that term can be taken away in the manner shown in p. 173.

in (4.); an assumption which, by destroying two terms in (3.), makes that equation become (5.). The expressions marked (6.) are obtained from (4.) and (5.); and, by taking four times the cube of the first of these two expressions from the square of the second, and extracting the square root, we get (7.), which by a slight modification becomes (8.). We then obtain (9.) and (10.) by taking half the sum and half the difference of (5.) and (8.), and extracting the cube roots of the results; and by adding these roots we get $y+z$, or

$$x=\sqrt[3]{\{-\tfrac{1}{2}b+\sqrt{[(\tfrac{1}{2}b)^2+(\tfrac{1}{3}a)^3]}\}}+\sqrt[3]{\{-\tfrac{1}{2}b-\sqrt{[(\tfrac{1}{2}b)^2+(\tfrac{1}{3}a)^3]}\}}\ \ldots\ (11.)$$

Since by (4.), $z=-\dfrac{a}{3y}$, we should also have

$$x=\sqrt[3]{\{-\tfrac{1}{2}b+\sqrt{[(\tfrac{1}{2}b)^2+(\tfrac{1}{3}a)^3]}\}}-\frac{a}{3y}\ \ldots\ (12.)^*;$$

where y is the first term. We saw (p. 199.) that the cube roots of 1 are 1, $\frac{1}{2}(-1+\sqrt{-3})$, and $\frac{1}{2}(-1-\sqrt{-3})$; and therefore we may have two additional values of y; by employing which, we should get by (12.) for the remaining roots of the equation

$$x=\tfrac{1}{2}(-1\pm\sqrt{3})\times\left(\sqrt[3]{\{-\tfrac{1}{2}b+\sqrt{[(\tfrac{1}{2}b)^2+(\tfrac{1}{3}a)^3]}\}}-\frac{a}{3y}\right).$$

We might also obtain equivalent expressions from (11.).

Exam. Required the roots of the equation $x^3-2x-4=0$.

Here, by (11.), $x=\sqrt[3]{\{2+\sqrt{(4-\frac{8}{27})}\}}+\sqrt[3]{\{2-\sqrt{(4-\frac{8}{27})}\}}$; or, by easy reductions, $x=\sqrt[3]{(2+\frac{10}{9}\sqrt{3})}+\sqrt[3]{(2-\frac{10}{9}\sqrt{3})}$: and from this, by performing the actual operations, we find $x=1{\cdot}577343+0{\cdot}422651=1{\cdot}999994$, or 2 nearly.†

By employing (12.) we should get $x=\sqrt[3]{\{2+(4-\frac{8}{27})\}}+\dfrac{2}{3y}$; whence we should obtain the same value for x, and rather more easily.

2. If we assume $p+q$, $p-q$, and $-2p$ as the roots of a cubic equation, that equation (§ 173.) will be $x^3-(3p^2+q^2)x+2p(p^2-q^2)=0$. Then, if we substitute $-(3p^2+q^2)$ for a, and $2p(p^2-q^2)$ for b, in $\sqrt{\{(\frac{1}{2}b)^2+(\frac{1}{3}a)^3\}}$, the radical which occurs in each of the values of x, we get $\sqrt{(-3p^4q^2+\frac{2}{3}p^2q^4-\frac{1}{27}q^6)}$, or, by an easy reduction, $(p^2-\frac{1}{9}q^2)\sqrt{-3q^2}$. Now (§ 121.) this expression will always be imaginary, when q is real; and when, therefore, all the roots, $p+q$, $p-q$, and $-2p$ are real. On the other hand, when q, and, therefore, the two roots $p+q$ and $p-q$ are imaginary, the same radical in the value of x is real. It thus appears,

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* This method of solution was first published by Cardan, an Italian mathematician, in 1545; and hence it is commonly called *Cardan's Method.* Mathematicians are agreed, however, that it was not discovered by him, but by two of his countrymen, Scipio Ferreus and Nicholas Tartaglia or Tartalea, independently of one another.

† The exact root is 2, the slight difference being occasioned by the inaccuracy in the last figures of the decimals.

that though Cardan's method always gives correct expressions for the roots of a cubic equation, it is useless in practice when the roots are all real, as we cannot then compute them by means of it, on account of the occurrence of the imaginary quantity. It is plain, that this will take place when a is negative, and $(\frac{1}{2}b)^2$ less in absolute magnitude than $(\frac{1}{3}a)^3$, or, which comes to the same, when $27b^2$ is less than $4a^3$. This has been called the *irreducible case;* and it is such, that all the attempts made by mathematicians to find any way of computing, by means of it, the roots of equations in which it occurs, have been unsuccessful. Cardan's method, therefore, is of no practical use, except when an equation has only one real root.*

3. Of the methods that have been devised for the resolution of an equation of the fourth degree, so that the roots may be expressed in terms of the given coefficients, that of Euler seems to be the simplest and most elegant. To investigate this method, let equation (1.) in the margin be the one to be resolved, and, as in (2.), let x be assumed equal to $\sqrt{p}+\sqrt{q}+\sqrt{r}$, where p, q, and r are quantities to be determined. Equation (3.) is got from (2.) by squaring. Then, by assuming $p+q+r=s$, as in (4.), transposing that quantity in (3.), and squaring, we get (5.), which becomes (7.) by means of the assumptions in (6.) and (2.). Equation (7.) will become identical with (1.) by the assumptions made in (8.); and from these assumptions we get (9.), (10.), and (11.), three equations which will enable us to find p, q, and r, and

$$x^4+ax^2+bx+c=0 \ldots\ldots\ldots\ldots\ldots (1.)$$

$$x=\sqrt{p}+\sqrt{q}+\sqrt{r} \ldots\ldots\ldots\ldots\ldots (2.)$$

$$x^2=p+q+r+2(\sqrt{pq}+\sqrt{pr}+\sqrt{qr}) \ldots (3.)$$

$$p+q+r=s \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

$$x^4-2sx^2+s^2=4(pq+pr+qr) + 8(\sqrt{p}+\sqrt{q}+\sqrt{r})\sqrt{pqr}. \quad (5.)$$

$$pq+pr+qr=t, \text{ and } pqr=u \ldots\ldots\ldots (6.)$$

$$x^4-2sx^2-8u^{\frac{1}{2}}x+s^2-4t=0 \ldots\ldots\ldots (7.)$$

$$-2s=a, \quad -8u^{\frac{1}{2}}=b, \quad s^2-4t=c \ldots\ldots (8.)$$

$$s, \text{ or } p+q+r=-\tfrac{1}{2}a \ldots\ldots\ldots\ldots\ldots (9.)$$

$$u^{\frac{1}{2}}, \text{ or } \sqrt{pqr}=-\tfrac{1}{8}b \ldots\ldots\ldots\ldots\ldots (10.)$$

$$t, \text{ or } pq+pr+qr=\tfrac{1}{16}(a^2-4c) \ldots\ldots\ldots (11.)$$

$$p^2-qr=-\tfrac{1}{2}ap-\tfrac{1}{16}(a^2-4c) \ldots\ldots\ldots (12.)$$

$$p^3+\tfrac{1}{2}ap^2+\tfrac{1}{16}(a^2-4c)p-\tfrac{1}{64}b^2=0 \ldots\ldots (13.)$$

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* Several expedients have been fallen upon for resolving cubic equations which belong to the irreducible case, such as the determination of the roots by approximation by means of series, the finding of them (see *Trigonometry*, 4th edition, § 224.) by means of cosines, &c. All such methods, however, except as matters of curiosity, are now superseded by Horner's method. The student who has a taste for matters of an antiquarian or historical kind will find much that is interesting on the subject here treated of, as well as on many others regarding algebra, in Hutton's *Mathematical Dictionary*, article ALGEBRA; and in his *Tracts*, vol. ii. He will find also a method of resolving cubic equations belonging to the irreducible case, by the late Mr. Barlow, by means of a table, in Barlow's *Mathematical Dictionary* and Rees's *Cyclopædia*, articles IRRE-

thence x. For effecting this, we eliminate q and r, which is done by multiplying (9.) by p, and taking (11.) from the product, so as to get (12.): then, by multiplying this by p, substituting, according to (10.), $-\frac{1}{8}b$ for pqr, and transposing, we get (13.), an equation which will give the value of p.

4. By a similar process we should find from (9.), (10.), and (11.) an equation with exactly the same coefficients for determining q and r, since all the three quantities, p, q, and r, are involved in the same manner in those equations. Hence, therefore, the three roots of (13.) must be the values of p, q, and r. Now we have seen (10.), that $p^{\frac{1}{2}}q^{\frac{1}{2}}r^{\frac{1}{2}} = -\frac{1}{8}b$. The signs, therefore, of $\sqrt{p}$, $\sqrt{q}$, and $\sqrt{r}$, the components of x, will depend on the sign of b, the sign of their product being contrary to its sign. Hence, when b is positive, either all the three, $\sqrt{p}$, $\sqrt{q}$, and $\sqrt{r}$, or only one of them, must be negative; while if b be negative, either all the three, or only one of them must be positive. By employing, therefore, all the varieties thus indicated, we shall have for the required roots,

For b positive.	For b negative.
$x = -\sqrt{p} - \sqrt{q} - \sqrt{r}$,	$x = \sqrt{p} + \sqrt{q} + \sqrt{r}$,
$x = -\sqrt{p} + \sqrt{q} + \sqrt{r}$,	$x = \sqrt{p} - \sqrt{q} - \sqrt{r}$,
$x = \sqrt{p} - \sqrt{q} + \sqrt{r}$,	$x = -\sqrt{p} + \sqrt{q} - \sqrt{r}$,
$x = \sqrt{p} + \sqrt{q} - \sqrt{r}$.	$x = -\sqrt{p} - \sqrt{q} + \sqrt{r}$.*

DUCIBLE CASE, and also in the third volume of Leybourn's *Mathematical Repository*, new series.

It may be remarked, that in some particular instances, the cube root of $-\frac{1}{2}b \pm \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}$ can be taken, when the radical part is imaginary. Thus, in the equation, $x^3 - 30x - 36 = 0$, we should have, by Cardan's formula (11.), $x = \sqrt[3]{(18 + 26\sqrt{-1})} + \sqrt[3]{(18 - 26\sqrt{-1})}$; the radicals in which have exact values, being equal to $3 + \sqrt{-1}$ and $3 - \sqrt{-1}$ respectively; and therefore the corresponding value of x is 6.

Exercises 1, 2, 3, and 8., p. 192., afford instances of the irreducible case.

* Were we to change $\sqrt{p}$ into $\frac{1}{2}\sqrt{p'}$, equation (13.), by this substitution, and by multiplication by 64, would become $p'^3 + 2ap'^2 + (a^2 - 4c)p' - b^2 = 0$; and the roots of this equation would be p', q', and r', where $\sqrt{q} = \frac{1}{2}\sqrt{q'}$, and $\sqrt{r} = \frac{1}{2}\sqrt{r'}$, Then, by combining these as above, we should have $x = -\frac{1}{2}(\sqrt{p'} + \sqrt{q'} + \sqrt{r'})$, $x = -\frac{1}{2}(\sqrt{p'} - \sqrt{q'} - \sqrt{r'})$, &c.

It may be remarked, that, in a practical point of view, this method of solution will fail, when equation (13.) or the corresponding one just given falls under the irreducible case of cubics; and there is a like failure in the other methods that have been proposed for the solution of equations of the fourth degree, except those by means of approximation, such as Horner's and others. When, indeed, the object in view is merely to obtain as near approximations as we please to the numerical values of the

NOTE C.

Sturm's Theorem.

THE following is Sturm's beautiful theorem for determining the number and positions of the real roots of equations.

Let $X=0$ be an equation which has not equal roots *, and in which $X=x^n+p_1x^{n-1}+\ldots\ldots+p_n$. Then (1.), by § 194., find the derived function which call X_1. (2.) To X and X_1 apply the rule given in § 81. for finding the greatest common divisor, merely varying the process by changing the signs of the final remainder obtained in dividing X by X_1, and calling the result, after this change, X_2: then by dividing X_1 by X_2, and calling the final remainder, with the like change of signs, X_3; and so on, till a remainder is found which does not contain x; the sign of which remainder is also to be changed. (3.) In each of the quantities X, X_1, X_2, &c., substitute for x a number α, and count the number m_1 of variations in the successive signs of the results. In the same quantities, again, substitute for x a number β, greater than α, and count the number m_2 of the variations in the signs as before. (4.) Then m_1-m_2 will be the number of real roots between α and β. (5.) If α be taken $=-\infty$, and $\beta=\infty$, m_1-m_2 will be the entire number of real roots belonging to the equation. (6.) If α be taken $=0$, and $\beta=\infty$, m_1-m_2 will be the number of the positive roots. The number of the negative ones will be found in a similar way by taking $\alpha=-\infty$, and $\beta=0$.

When x is assumed $=\infty$, in any of the quantities X, X_1, &c., the sign of the result is the same as the sign of the first term of that quantity, the other terms vanishing in comparison of that one. The same is the case when $-\infty$ is employed, except that the signs of the odd powers of x are to be changed. When x is taken $=0$, the functions X, X_1, &c., are reduced to their last terms.

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incommensurable roots of equations of the third and higher orders, Horner's method is far preferable to any other; and Cardan's, Euler's, and others of a similar kind, are little else than analytical curiosities, possessing some interest. It may be remarked, in conclusion, that all the efforts of algebraists have failed in discovering any other method than that of approximation for resolving equations of a higher order than the fourth, except in a few particular cases. (See pp. 192—199.)

By applying the methods established in this Note to some of the examples and exercises given in Chapter X., the student will feel, in a striking degree, the superiority of Horner's method in computing the roots of equations.

* Should any proposed equation have equal roots, let them be determined by § 195. Then (§ 174.), by dividing the given equation by the proper factor, an equation will be obtained such as the one assumed in the text.

The following is an outline of a proof of this theorem:—

I. In a series of the form, $Aa^nh + Ba^{n-1}h^2 + Ca^{n-2}h^3 +$ &c., or, which is the same, $h(Aa^n + Ba^{n-1}h + Ca^{n-2}h^2 + \&c.)$, h may be taken so small that the value of the entire series will have the same sign as its first term: for the first term within the vinculum is independent of h, while each of the others may be reduced to any degree of minuteness whatever by the continued diminution of that quantity.

II. If q_1, q_2, &c. be the several quotients obtained in operating on X and X_1 according to the rule, we have, by the nature of the process,

$$X = X_1q_1 - X_2,\ X_1 = X_2q_2 - X_3,\ \ldots,\ X_m = X_{m+1}q_{m+1} - X_{m+2}.$$

From these expressions it is plain, that no value whatever of x can reduce to 0 any two consecutive ones of the functions X, X_1, &c.: for, if two of them, suppose X_1 and X_2, could vanish simultaneously, it would appear from the second of the foregoing equations, and from those following it, that all the succeeding functions, $X_3, X_4, \ldots, X_m$ would also vanish, which cannot take place, as X_m is a *number* independent of x.

III. If a value be assigned to x which shall make one or more of the functions *after* X vanish, the signs of the two functions immediately preceding and following the vanishing one are opposite. Thus, if $X_2 = 0$, the second equation in II. gives $X_1 = -X_3$. Hence, putting 0 between the signs of X_1 and X_3, we have either +, 0, −, or −, 0, +; which, whether the sign of 0 be taken as + or −, will give one variation of signs, and only one, and will not affect, therefore, the entire number of variations in the signs of X, X_1, &c. Hence, no change in the value of x will affect the number of the variations in the signs of those functions, unless x be taken equal to a root of $X = 0$, a case which is now to be considered.

IV. Let, then, a be a root of $X = x^n + p_1x^{n-1} + \ldots + p_n = 0$; and let h be a quantity so small, that between $a - h$ and $a + h$ there is no root of $X = 0$ except a: then X and $X_1 (= nx^{n-1} + (n-1)p_1x^{n-2} + \ldots + p_{n-1})$ have the same signs when x is taken equal to $a + h$, but opposite signs when it is taken $= a - h$. To prove this, let $qx^r + b = 0$ have a as a root, so that $qa^r + b = 0$. Then, if in $qx^r + b$, and in the derived function qrx^{r-1}, we substitute $a + h$ for x, we get (§ 210.), by rejecting $qa^r + b = 0$,

$$q\{ra^{r-1}h + \frac{r(r-1)}{1\cdot 2}a^{r-2}h^2 + \&c.\}, \text{ and } qr\{a^{r-1} + (r-1)a^{r-2}h + \&c.\}.$$

Now (I.) h may be taken so small, that, in each of these results, the sign of the whole will be the same as the sign of the first term; and those first terms will evidently have the same signs when h is positive, but opposite signs when it is negative. It is plain, also, that X is made up of quantities, such as $qx^r + b = 0$, and of no others; and therefore the property proved regarding $qx^r + b = 0$ must hold regarding $X = 0$.

V. It follows from III. and IV., that, if a be a number which is nearer $-\infty$ than any of the roots of the equation $X = 0$, and if successive increasing values greater than a be substituted for x, one variation, and only one, in the signs of X, X_1, &c., is lost, and one permanence gained, whenever any of these values is a root of the equation $X=0$; and hence the truth of the theorem is manifest.

Sturm's own proof will be found in the *Mémoires de l'Institut de France* for 1835; see also Fourcy's *Algèbre*, and Young on *Equations*. In the last-mentioned work, various important contractions and simplifications in the practical application of the rule are pointed out; and much advantage will be obtained by employing the method of detached coefficients.

As an example, let it be required to find the number and limits of the real roots of the equation, $x^3-4x^2-4x+20=0$. (See p. 183.)

Here $X=x^3-4x^2-4x+20$, and therefore (§ 194.) $X_1=3x^2-8x-4$. Then, for avoiding fractions in employing § 81., we multiply X by 3; and, after dividing the product by X_1, we divide the remainder by 4 for a simplification; and, for avoiding fractions, we multiply the quotient by 3. Continuing the division by X_1, we get as remainder $-14x+41$; and therefore $X_2=14x-41$. In dividing X_1 by this, to prevent fractions from arising, we multiply X_1 and the first remainder each by 14; and we obtain as final remainder -333. Hence we have $X_3=333$. It is plain that the multiplication by 3 above referred to, the division by 4, the multiplication again by 3, and the two multiplications by 14, are allowable; as the *signs* of the quantities here denoted by X_2 and X_3 are evidently the same as those that would have been obtained by changing the signs of the successive remainders, had no such multiplications or divisions been employed.

Now, when x is taken first equal to $-\infty$, and then to ∞, in X, X_1, &c., the signs of the results are, respectively,

$-+-+$, where there are three variations, and
$++++$, where there is no variation.

Hence $m_1-m_2=3-0=3$: the equation, therefore, has three real roots. By taking, also, $x=0$, the signs are found to be

$+--+$, where there are two variations.

The equation will have, therefore, one $(=3-2)$ negative root, and two $(=2-0)$ positive ones.

Now, for ascertaining the positions of the roots within narrow limits, if we take $\alpha=0$ and $\beta=10$ in X, X_1, &c., we find that there are two roots between these limits: and it would appear, in a similar manner, that there are two between 0 and 5. By taking, however, $\alpha=0$, and $\beta=3$, it is found that there is but one root between these limits, so that between 3 and 5 there must be another. By other substitutions we readily find that the two positive roots lie between 2 and 3, and between 3 and 4: and if we chose to determine still narrower limits, it would be easily shown that those roots would lie between 2·6 and 2·7, and between 3·5 and 3·6. It would appear, in a similar way, by taking $\alpha=-10$, $\beta=0$, &c., that the negative root would lie between -2 and -3.

Note D.

On Inaccuracies in Style which frequently occur in Mathematical Composition.

It may not be improper here to caution learners against some inaccuracies in style and expression which are often met with in scientific works published in this country.* To matters of this kind much attention is paid by the French mathematicians. By this means, as may naturally be expected, accuracy is produced in thought as well as in style; and hence it is, that many of their recent works exhibit a perspicuity and simplicity, and a logical clearness of arrangement, which enhance their value in a very considerable degree. To put the student on his guard against faults of this kind, some instances of the inaccuracies above referred to are subjoined; and he will find that, by avoiding the use of them and of similar modes of expression, he will soon acquire a comparatively correct style, instead of the careless and random one that is so common in this country. Some of these instances will doubtless be found in the works of authors of the present day, but they have been purposely selected from the writings of authors no longer living.

"Place the two quantities under each other." This is impossible. *One* of the quantities may be under the other, but *each* cannot be under the other simultaneously. It should be specified which is to be put below the other. "Subtract the numerators from each other," presents a like inaccuracy.—The following contains two or three improprieties, the word *divide* being used in a wrong sense in the second instance: "Divide the numerators by each other, if *they* will exactly divide."—"The square root of any quantity may be either + or −." This should be *the sign of the square root of any quantity may be either + or −*, or *the square root of any quantity may be either positive or negative.*—"An equation of the third degree or power." No equation is of the third or any other *power*. The expression "or power," should be omitted.—"To find the root of the equation, $x^3-15x^2+63x=50$." It should be *a root* or *the roots*, and not *the root*, as every cubic equation has three roots; and Hutton himself, from whom this is taken, computes all the three.—"Powers of the same quantity are divided by subtracting the exponent of the divisor from that of the dividend; the remainder is the exponent of the quotient." This is faulty in more respects than one. It is not *powers* of the same quantity that are divided, but one power of it is divided by another; and, at the conclusion, it is not stated to what the exponent of the quotient belongs.—"*The* power [it should be *a* power] of a quantity is its square, cube, biquadrate, &c.; called also its second, third, fourth power, &c." This miserable attempt at a definition is from Bonnycastle.*—"A term is any part or member of a compound quantity which is separated from the rest by the signs [it should be *sign*] + or −." Here the words, *part* and *member* are used in no definite or accurate sense.—"If the unknown quantity be in the form of a surd, let it be made to stand alone," &c. The unknown quantity may be *contained* or *involved* in a surd, but it can itself be in no *form* but that of *x*, *y*, or some other symbol assumed to represent it.—"Neither the first nor the last terms are squares," ought to be, "neither the first nor the last term *is a square*." The like inaccuracies occur in the following expression: "The first or last *terms* of the expression *are cubes*."—"This value is the greatest of all others." Here *others* ought to be omitted, or it might be, "this value is greater than any other." "The greatest of any" is also faulty. It should be "greater than any," or "greatest of all."—The expression, "general case," which is used by both English and foreign writers, is of very questionable propriety. A *particular* case of a problem or investigation is quite correct and intelligible, but to speak of a *general case* seems to involve a contradiction of terms.

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* "Levia quidem hæc, et parvi fortè, si per se spectentur, momenti. Sed ex elementis constant, ex principiis oriuntur, omnia: et ex judicii consuetudine in rebus *minutis* adhibitâ, pendet sæpissimè etiam in *maximis* vera atque accurata Scientia."—*Dr. Samuel Clarke.*

In geometry, the expression, "circumscrib*ing* circle," should be laid aside, and be replaced by the correct analogical one, "circumscrib*ed* circle." If we revert to the etymology of the words, "*inscribing* circle" would be no more incorrect than "*circumscribing* circle;" the former of which, however, the most negligent writer would never employ. The "*inscribed* circle" is the circle described *in* a figure; the "*circumscribed* circle" is the one described *about* it.—"To raise a perpendicular," "to erect," "to let fall," "to demit," and "to drop" one, are all objectionable expressions, as none of them can be used with any propriety, except when the perpendicular lies *above* the other line. The expression "to *draw* a perpendicular" is always correct, and it alone ought to be employed.—The expression "the angle A is equal to the angle B, being right angles," should be "the angle A is equal to the angle B, *each of them being a right angle*," or "*the angles* A *and* B *are equal*, being right angles."

The following instances occur, among many similar ones, in Leslie's *Geometry of Curve Lines*. "The point *shoots into the indefinite distance*." "The point *vanishes into extreme remoteness*." "This rectangle *melts* into the elementary space BM*mb*, while the ordinate *bm migrates* into BM, and the secant *passes* into a tangent." "The cissoid received its name from the Greek word for ivy, because it appears *to mount along its asymptote* in the same manner as that *parasite plant climbs on the tall trunk of the pine*." These sentences exhibit an aim at puerile ornament, at injudicious variety of expression, and at oratorical language, which is totally unsuited to a work on abstract science, and altogether inconsistent with good taste. In mathematical reasoning, all is addressed to the judgment, and nothing to the passions or feelings; and therefore, in such reasoning, every expression of a figurative or purely ornamental nature is to be rejected; the excellence and the true beauty of mathematical composition consisting in its simplicity, in its freedom from every thing superfluous, and in having every word used in its natural and appropriate meaning.

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* This author often employs long sentences devoid of unity and natural connexion. Thus, in the Introduction to his larger *Algebra* several of his sentences fill more than half an octavo page, and are composed of ill-assorted subordinate parts, awkwardly tacked together by repetitions of *who*, *which*, *that*, *as*, *when*, *since*, and other connecting words.

Some mathematical writers are fond of addressing their readers in the second person: thus, "by multiplying by $x-a$, *you* will get," &c. This is always inelegant, except in the single case in which the imperative mood, almost deprived of its imperative character, is used for convenience and brevity in such expressions as "join AB," "divide by a," &c. According to good usage, the writer may employ the first person plural when he states the result at which he and his readers arrive by means of the process which is followed; thus, "we obtain by transposition," &c. Such phraseology, however, ought not to be used when he speaks of himself individually. Thus, "we are of opinion" (meaning that it is the opinion of the author) should be "I am of opinion," "it seems probable," or the like.

Many other instances of careless and incorrect modes of expression might be adduced. Those, however, that have been adverted to above, will turn the attention of the intelligent student to the subject, and will make him feel that, while accuracy in reasoning is the thing mainly to be attended to in his investigations, yet correctness and elegance of expression in the communication of his ideas are matters of much importance. He should recollect also, that, while no person has ever acquired a faultless style, as little has any one succeeded in writing even moderately well without studying with care and attention the means to be pursued for accomplishing the object.

THE END.

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ERRATA.

Page 47. line 4. from bottom, for "$-r$" read "$1-r$."

80. The cube root of 7 should be 1·912931.

" 21 " 2·758924.

" 30 " 3·107233.

193. line 3., for "p_{-1}" read "p_{n-1}."

261. line 17., for "(*Euc.* I. 31.)" read "(*Euc.* III. 31.)."

284. line 16., for "$\sqrt{3}$" read "$\sqrt{-3}$."

286. line 4., for "$-\frac{1}{8}b$" read "$\frac{1}{64}b^2$."

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